



Grade 7/8 Math Circles

February 17/18, 2015

Circle Geometry Solutions

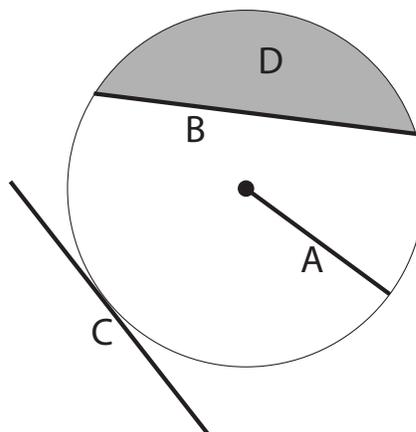
Circles: They're Not Pointless

To be mathematically accurate, you could indeed argue that circles are “pointless” because, well, they have no points! However, circles are arguably one of the most important fundamental shapes, besides triangles. Today’s lesson flows naturally from last week’s topic of π . We’ll be discussing important terminology, properties, and theorems. You’ll also have the opportunity to try a hands-on activity.

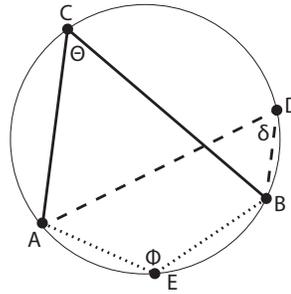
Warm-Up

Try to answer the following 8 questions in 15 minutes without a calculator. Don't worry if you can't answer them all. You'll be an expert by the end of this lesson!

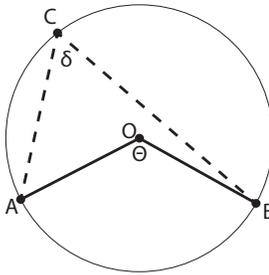
1. What is the proper term for the line A on the diagram below? **radius**
2. What is the proper term for the line B on the diagram below? **chord**
3. What is the proper term for the line C on the diagram below? **tangent**
4. What is the proper term for the section D on the diagram below? **segment**



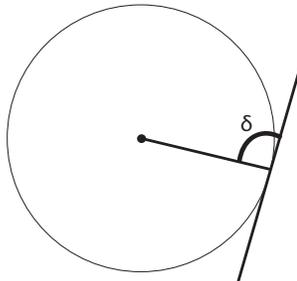
5. If the angle θ on the diagram below is 37° , what is the angle δ ? What is the angle ϕ ?
 $\delta = 37^\circ$ (ASSAT), $\phi = 180 - 37 = 143^\circ$ (CQT)



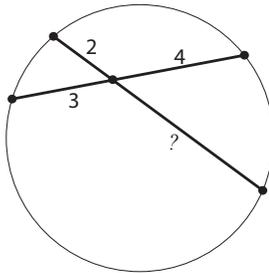
6. If the angle θ on the diagram below is 110° , what is the angle δ ? $110 \div 2 = 55^\circ$ (STT)



7. What is the angle δ in the diagram below? 90° (TRT)



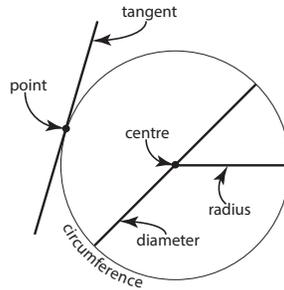
8. What is the missing line segment length in the diagram below?
 $3 \times 4 = 12$; $12 \div 2 = ? = 6$ (CCT)



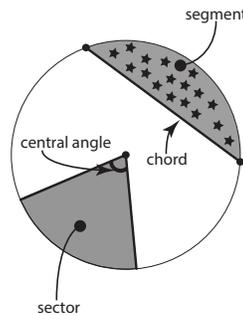
Terminology

Warm-Up (WU) 1 through 4 tested your knowledge of circle terminology. This section provides you with all the terms you need to know in order to understand the rest of the lesson.

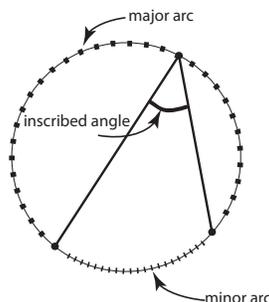
The diagram below depicts four terms you should already know: *circumference*, *centre*, *radius*, and *diameter*. There are also a couple of new concepts on this diagram. For our purposes, a *point* is any location on the circumference of a circle. A *tangent* is a line that passes through only one point on a circle's circumference.



A *sector* is a portion of a circle trapped by two radii (plural of radius). A *central angle* is an angle whose vertex is the centre of a circle and whose sides are radii intersecting the circle in two distinct points. We say the central angle is *subtended* by the *arc* (section of the circumference) between the two distinct points. A *chord* is a line segment that connects two distinct points of a circle. A *segment* is a portion of a circle made by a chord and an arc between the two endpoints of the chord.

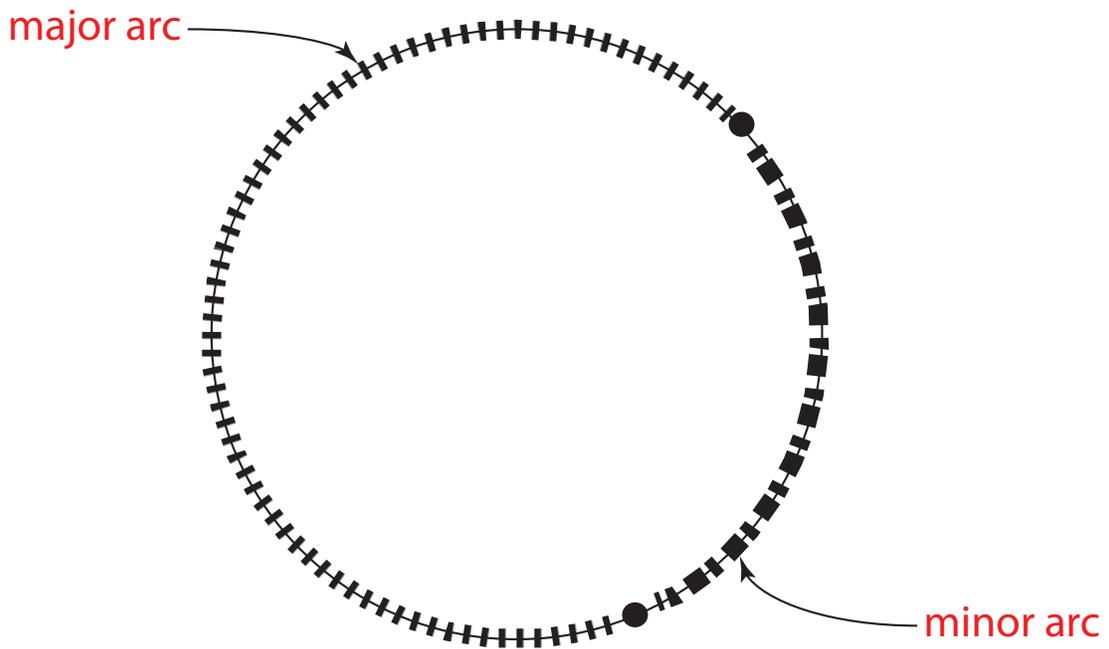
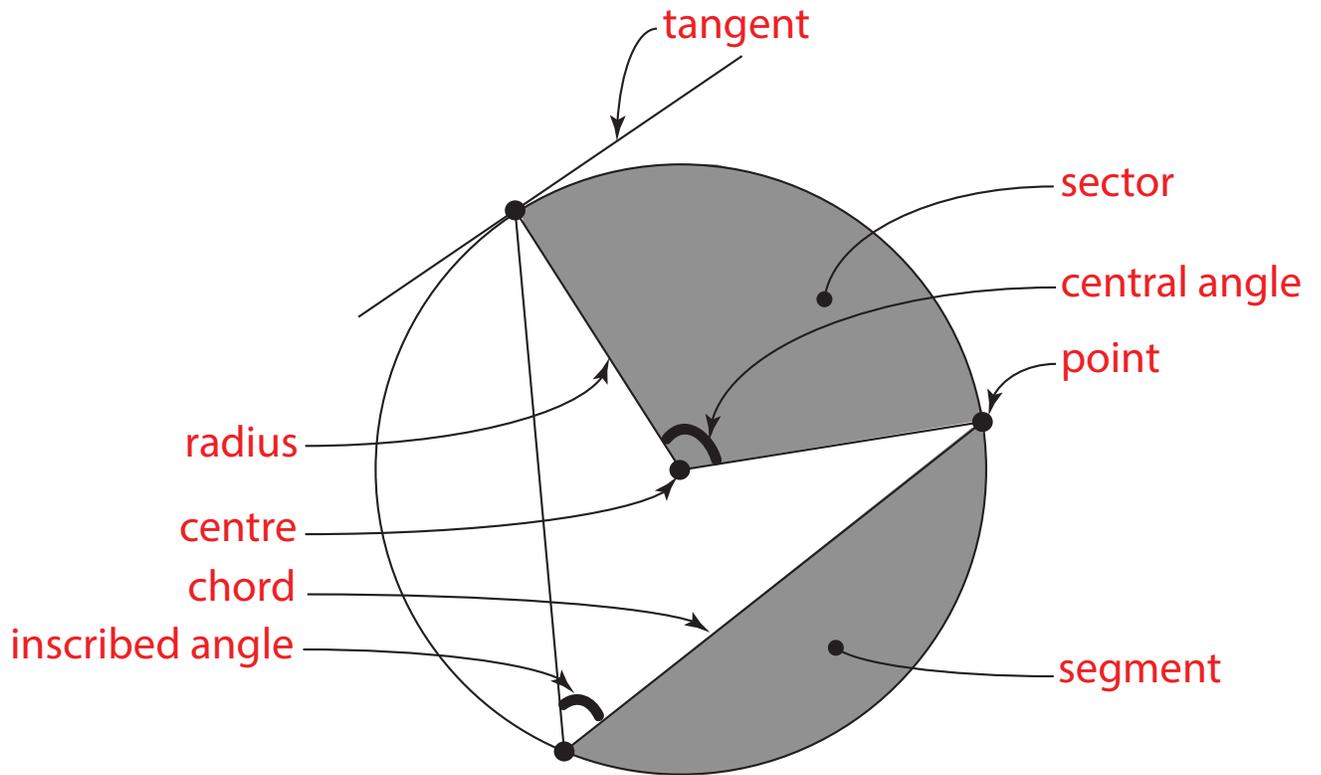


A *major arc* is the longer arc joining two points on the circumference of a circle. A *minor arc* is the shorter arc joining two points on the circumference of a circle. An *inscribed angle* is an angle formed by two chords in a circle which have a common endpoint.



Try it out:

Label the diagrams below.

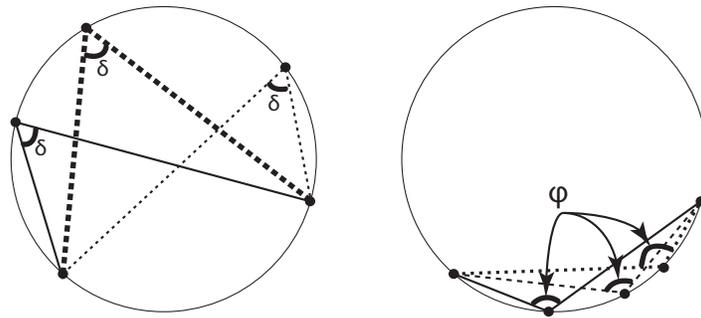


Inscribed Angle Theorems

WU 5 and 6 tested your knowledge of theorems involving inscribed angles. In this section, we will look at these interesting theorems.

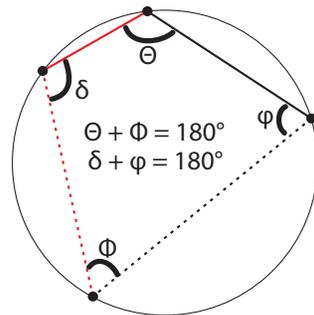
Angles Subtended by the Same Arc Theorem (ASSAT):

An inscribed angle is always the same along the same arc where the endpoints are fixed.



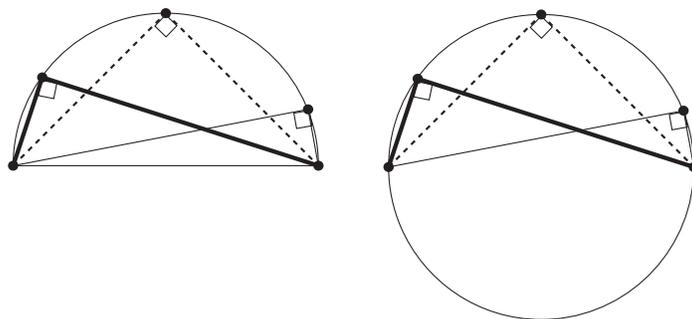
Opposite Inscribed Angles Theorem/Cyclic Quadrilateral Theorem (CQT):

Opposite inscribed angles (on opposite arcs) always add up to 180° .



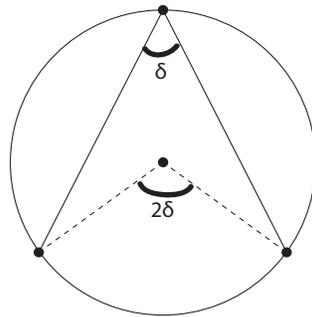
Angle in a Semicircle Theorem (AST):

An angle inscribed in a semicircle (i.e. the endpoints are at either end of the diameter) is always a right angle.



Central Angle Theorem/Star Trek Theorem (STT):

An inscribed angle is half of the corresponding central angle.

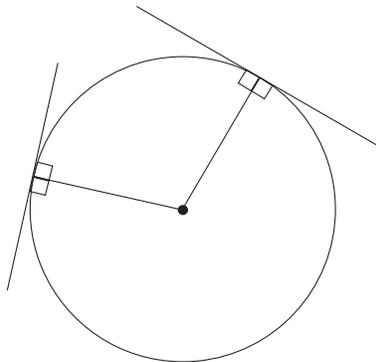


Tangent Theorem

WU 7 tested your knowledge of a theorem involving the tangent to a circle.

Tangent-Radius Theorem (TRT):

If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of tangency.

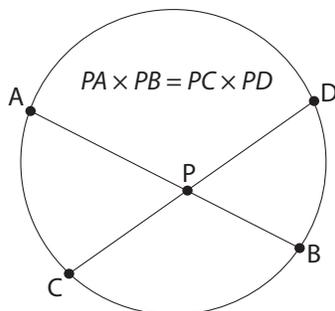


Chord Theorem

WU 8 tested your knowledge of a theorem involving chords.

Crossed Chord Theorem (CCT):

If two chords intersect inside a circle then the product of the lengths of the segments of one chord equals the product of the lengths of the segments of the other chord.



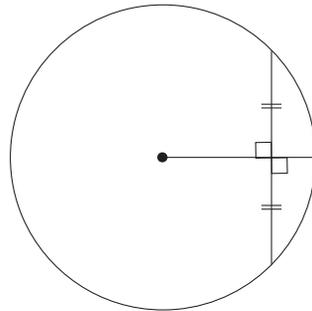
Perpendicular Bisectors

There is a very useful three-part theorem that relates chords and radii. Before we get to it, we need to know some more terminology.

Bisect means to divide into two equal parts. The dividing line is called a *bisector*. You should know that *perpendicular* means at a 90° angle. A *perpendicular bisector* is a line which cuts a line segment into two equal parts at 90° .

Radius as a Perpendicular Bisector Theorem (RPBT):

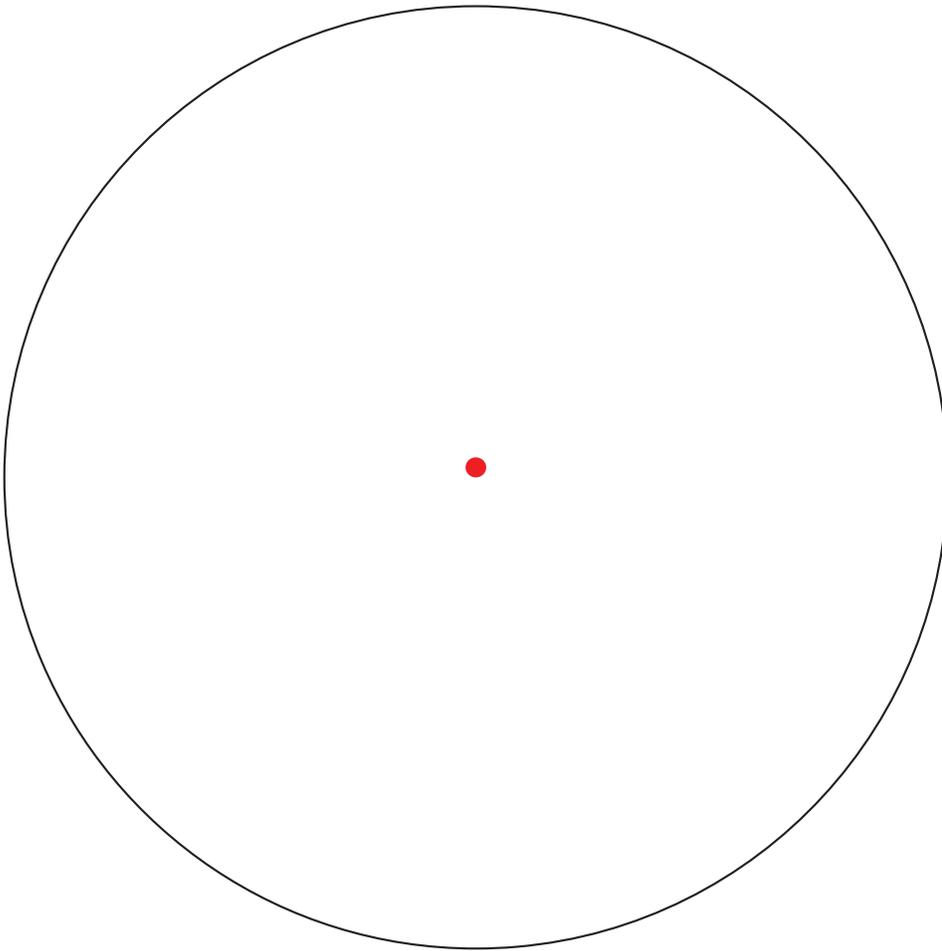
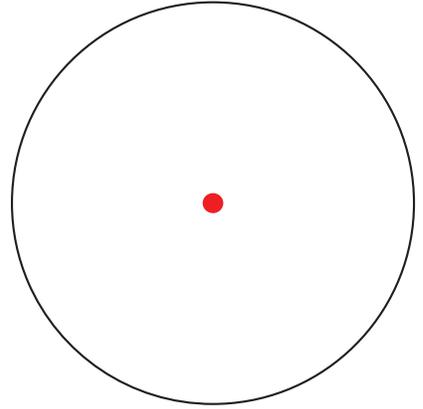
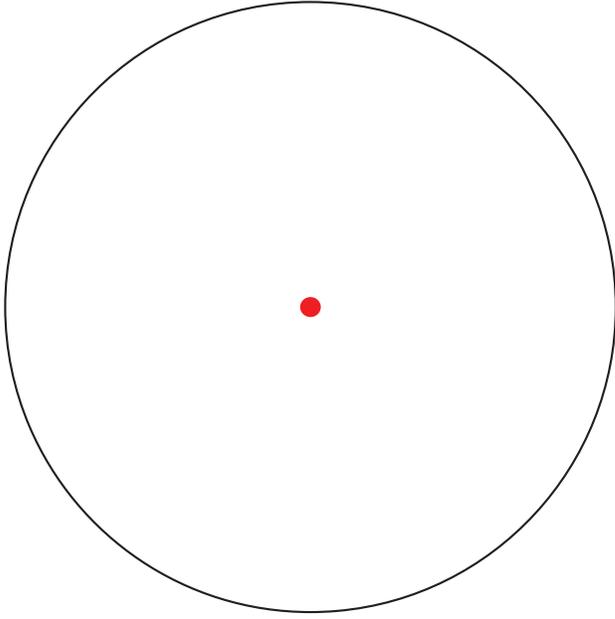
1. In a circle, a radius that is perpendicular to a chord bisects the chord.
2. In a circle, a radius that bisects a chord is perpendicular to the chord.
3. In a circle, the perpendicular bisector of a chord passes through the centre of the circle.



Activity: Finding the centre of a circle

1. Given a circle, draw any two chords using a ruler. Make sure they are not parallel.
2. For both chords, you are going to construct the perpendicular bisector:
 - (a) Place the compass on one end of the chord.
 - (b) Set the compass' width to approximately two thirds the chord length.
 - (c) Without changing the compass' width, draw an arc above and below the chord.
 - (d) Again without changing the compass' width, place the compass' point on the the other end of the chord. Draw an arc above and below the chord so that the arcs cross the first two.
 - (e) Using a ruler, draw a line between the points where the arcs intersect. This is the perpendicular bisector of the chord.
3. According to the theorem above, the point where the perpendicular bisectors intersect must be the centre of the circle.

Try it out on the next page.



Wrap-Up

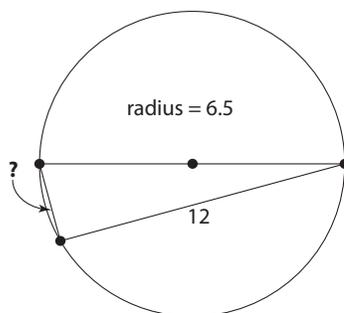
Today you learned about one of the most important and interesting two-dimensional shapes. You learned a lot of new terms and quite a few properties of circles. Many of the theorems you've learned in this lesson can be connected with the Pythagorean Theorem to create problems involving both triangles and circles.

Problem Set

Complete the following problems without a calculator. **State each theorem as you use it.** You may find the Pythagorean Theorem useful for some of the problems.

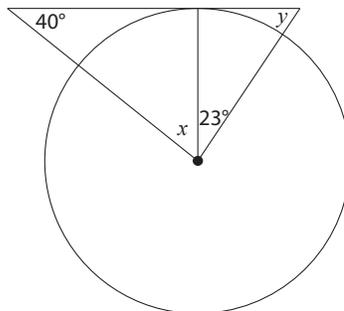
1. Redo the warm-up. See if you can answer more questions than before!
2. Find the length of the missing triangle side below.

By AST, we have a right-angled triangle. The hypotenuse is the diameter, which is 13. Then by Pyth Thm, the missing side length is $\sqrt{13^2 - 12^2} = 5$.

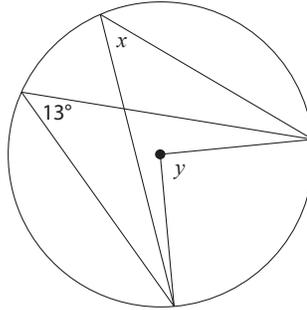


3. Find the missing angles x and y below.

By TRT, we have two right-angled triangles. Recall the interior angles of a triangle add to 180° . Thus, $x = 180 - 90 - 40 = 50^\circ$ and $y = 180 - 90 - 23 = 67^\circ$.

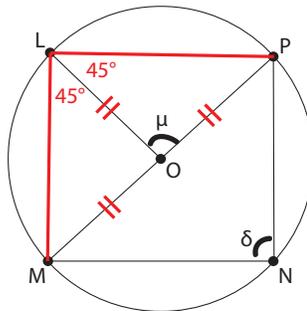


4. Find the missing angles x and y below. By ASSAT, $x = 13^\circ$. By STT, $y = 13 \times 2 = 26^\circ$.



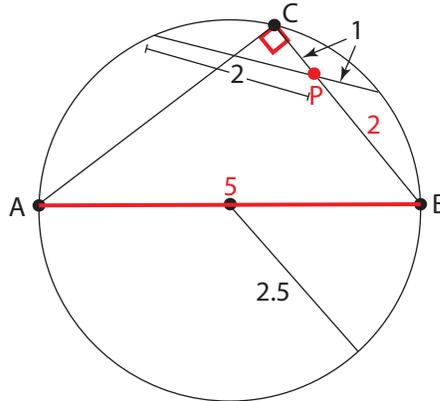
5. Given that MP is a diameter and μ is 90° , show that the angle δ below is 90° *without using the Angle in a Semicircle Theorem or the Star Trek Theorem*. Make sure to support any assumptions you make.

If we connect M to L and P to L, we create a cyclic quadrilateral. You cannot immediately assume it is a square, though it certainly looks like one, thanks to the diagram. Since OM , OP , and OL are all radii, they are equal. So we have two isosceles triangles, MOL and LOP . Since μ is 90° , angle OLP is $(180 - 90) \div 2 = 45^\circ$. Using the same logic, angle OLM is also 45° . Then, angle MLP is 90° . Finally, by CQT, δ must be $180 - 90 = 90^\circ$.



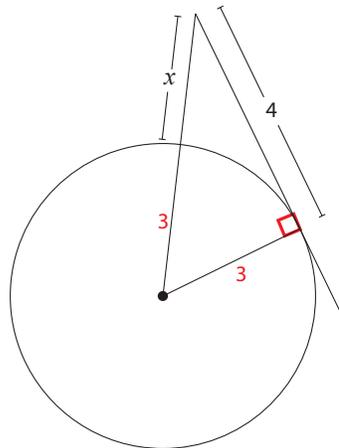
6. Find the length of chord AC below. Assume A and B lie on the diameter.

Let the intersection of the chords with given lengths be P . By CCT, we see that PB is 2. Thus, $BC = BP + PC = 3$. Now, since the radius is 2.5 and AB is a diameter, we get that AB is 5. By AST, we see that triangle ABC is right-angled. Then by Pyth Thm, we find that $AC = \sqrt{5^2 - 3^2} = 4$.



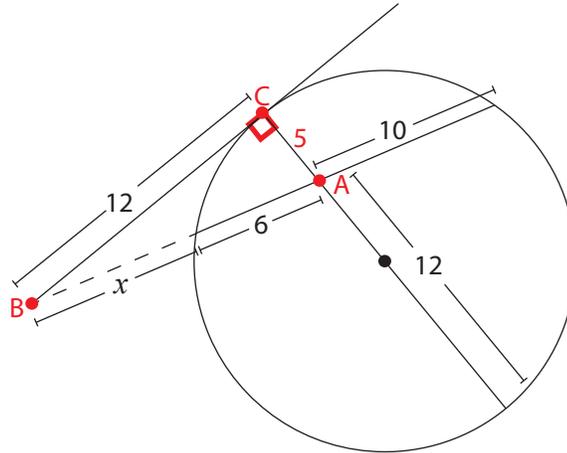
7. Given that the radius of the circle below is 3, find the length of the line segment x .

By TRT, we have a right-angled triangle. Then by Pyth Thm, the hypotenuse is $\sqrt{4^2 + 3^2} = 5$. Then, $x = 5 - 3 = 2$.



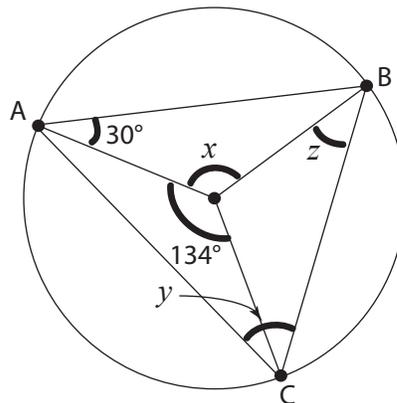
8. What is the length of the dashed line segment x below?

Labelling the points as below, by CCT, we see that AC is 5. By TRT, triangle ABC is right-angled. By Pyth Thm, we find that AB is $\sqrt{12^2 + 5^2} = 13$. Then, $x = 13 - 6 = 7$.



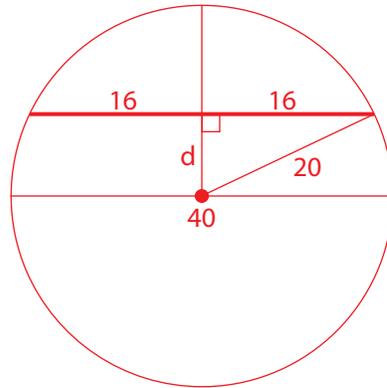
9. Determine the missing angles x , y , and z below.

Since AB , BC , and AC are all radii, we have three isosceles triangles. This means that $x = 180 - 30 - 30 = 120^\circ$. Then, by STT, $y = 120 \div 2 = 60^\circ$. Since a circle is 360° , the bottom right sector's angle is $360 - 120 - 134 = 106^\circ$. Then, since we have an isosceles triangle, $z = (180 - 106) \div 2 = 37^\circ$.



10. A circle has diameter 40. A chord of length 32 is drawn parallel to the diameter. What is the distance between the diameter and the chord?

Drawing a diagram, we can use RPBT to create a right-angled triangle. We are looking for d , the distance between the chord and the diameter. By RPBT, the radius perpendicular to the chord splits it into two equal halves of 16. Then, draw a radius to create the hypotenuse of a right-angled triangle. Then, by Pyth Thm, $d = \sqrt{20^2 - 16^2} = 12$.



11. Find the missing angle x below.

Hint: Recall that when two lines cross, the opposite angles are equal.

While it may not be visually obvious, we can use STT, as B is on the major arc of AC . By STT, angle AOC is $2x$. Using the hint, we can label the unlabelled angle in both triangles as y . This gives us $180 = 2x + 16 + y$ and $180 = x + 37 + y$. Since both equations are equal to 180, they are equal to each other. So we have $2x + 16 + y = x + 37 + y$. We begin to "balance" this equation by subtracting y from both sides. This gives us $2x + 16 = x + 37$. We now subtract x from both sides, resulting in $x + 16 = 37$. We finish off balancing our equation by subtracting 16 from both sides, resulting in our final answer, which is $x = 21^\circ$.

