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Grade 7/8 Math Circles

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Variables in Real Life: Kinematics *Solutions*

The Science of Motion

Today, we're going to use variables (also known as unknowns) to solve real life problems. We're going to take a look at a branch of the physical sciences called *kinematics*. Kinematics is the study of motion. The three main quantities that we are going to use in this lesson are *displacement*, *velocity*, and *time*. Let's get moving!

Warm-Up

Try to answer the following 7 questions in 9 minutes. You may use a calculator. Don't worry if you can't answer them all. You'll be an expert by the end of this lesson!

1. What are the standard units of displacement, velocity, and time?
   displacement - metres, velocity - metres per second, time - seconds

2. Convert 100 m/s to km/hr. $100 \times 3.6 = 360$ km/hr

3. Ryan is dribbling a soccer ball at 2 m/s. The 6-yard box is 100 m away. How long does it take him to run to the 6-yard box? $t = d \div v \implies t = 100 \div 2 = 50$ seconds

4. Ryan kicks the soccer ball at 10 m/s. The ball crosses the goal line after 0.55 s. How far away did Ryan kick the ball from? $d = v \times t \implies d = 10 \times 0.55 = 5.5$ metres

5. Ryan takes a penalty kick from 11 m away from goal. The ball takes 2.2 s to cross the goal line. How fast was the ball travelling? $v = d \div t \implies v = 11 \div 2.2 = 5$ m/s

6. Sachin is a faster runner than Ryan. To be polite, he gives Ryan a 2 second head start in a 100 m dash. If Sachin runs at 5 m/s and Ryan runs at 4 m/s, who will win the race? We must find how long it takes each person to run 100 m using $t = d \div v$. We find that it would take Sachin 20 s and Ryan 25 s. But we must add 2 seconds to Sachin's time because Ryan has a head start (assuming we start our clock when Ryan starts running). So Sachin takes 22 s to cross the line from when Ryan starts. But since Ryan takes 25 s to cross the line, Sachin will win the race.

7. How long after Ryan started running did it take Sachin to catch up?
   After drawing a picture, we find that the key relationships are $d_R = d_S = d$ and $t_S = t_R - 2$. Use these relationships and the velocities given:
   $4t_R = 5 \times (t_R - 2)$
   $4t_R = 5t_R - 10$
   $10 = t_R$
   $\therefore$ Sachin catches up to Ryan 10 seconds after Ryan starts running.

The Usual Quantities

As mentioned before, the three main quantities we will be using are displacement, velocity, and time. For the time being, you can consider displacement to be the same as distance, and velocity to be the same as speed. You will learn the difference between these quantities in high school.

To solve all of the problems in this lesson, we need to know the relationship between these quantities. Let us call displacement d, (average) velocity v, and time t. The relationship is:

$$v = d \div t$$

Standard Units

Warm-Up (WU) 1 tested your knowledge of standard units. In scientific calculations, there is a set of internationally agreed-upon units that mathematicians and scientists use. These are known as SI units or standard units. When you are making calculations, you should always convert to SI units. The SI units are as follows:

- displacement - metres (m)
- velocity - metres per second (m/s)
- time - seconds (s)

Converting Velocities

WU 2 tested your ability to convert velocities. You should know how to convert displacements and times using the table below.

1 kilometre (km)	1000 metres (m)
1 metre	100 centimetres (cm)
1 centimetre	10 millimetres (mm)
1 metre	0.001 kilometres
1 hour (hr)	60 minutes (min)
1 minute	60 seconds (s or sec)

Converting velocities is not as simple. We must create a *conversion factor*. The idea behind our conversion factor is that we want to "cancel" our given units and end up with the standard units.

Let's take a look at how to convert from m/s to km/hr:

per

$$\begin{aligned}\frac{\text{1 metre}}{\text{1 second}} &= \frac{1\ \cancel{\text{metre}}}{\text{1 second}} \times \frac{\text{1 kilometre}}{1000\ \cancel{\text{metres}}}\\ &= \frac{\text{1 kilometre}}{1000\ \cancel{\text{seconds}}} \times \frac{60\ \cancel{\text{seconds}}}{1\ \cancel{\text{minute}}} \times \frac{60\ \cancel{\text{minutes}}}{\text{1 hour}}\\ &= \frac{\text{3600 kilometres}}{\text{1000 hours}}\\ &= 3.6 \times \frac{\text{1 kilometre}}{\text{1 hour}}\end{aligned}$$

Thus, 1 m/s = 3.6 km/hr. So, to go from m/s to km/hr, you multiply the quantity by 3.6. Through a similar process, you will find that to go from km/hr to m/s, you should divide the quantity by 3.6.

Try it out:

Convert 3 km/min to m/s. 50 m/s

$$\begin{aligned}\frac{\text{3 km}}{\text{1 min}} &= \frac{3\ \cancel{\text{km}}}{\text{1 min}} \times \frac{\text{1000 m}}{1\ \cancel{\text{km}}}\\ &= \frac{\text{3000 m}}{1\ \cancel{\text{min}}} \times \frac{1\ \cancel{\text{min}}}{\text{60 sec}}\\ &= \frac{\text{3000 m}}{\text{60 sec}}\\ &= 50 \times \frac{\text{1 m}}{\text{1 sec}}\end{aligned}$$

Approaching Kinematics Problems

WU 3 through 7 are typical kinematics problems. The rest of the lesson focuses on how to solve these problems. The key is the relationship between the three quantities.

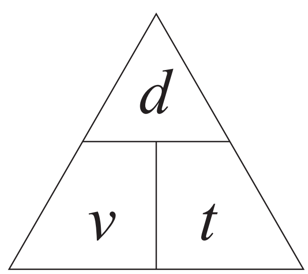
In every problem, you will be given information about at least two of the three quantities. Follow these steps to solve the problem:

1. Always start by writing down what you are **given** in standard units.
2. Next, write down what you are **required** to find.
3. Then, you **apply** your knowledge by writing down the equation that states the relationship of the three quantities.
4. Now, you **solve** the equation using what you are given to find what is required.
5. Lastly, you write a **statement** declaring the answer.

This is called the **GRASS** method of solving problems.

Solving the Problem

Step 4 of the GRASS method is the most difficult. You need to be able to rearrange the equation that gives the relationship of the three quantities. There are two ways to solve your equation. The more rigorous (mathematically proper) way to do this is to rearrange the equation to solve for what you want. The other way is through a trick called the kinematics triangle.



To use the triangle:

1. Circle the quantity that you are required to find.
2. Find the letters which correspond to the quantities you are given.
   (a) If they are **beside** each other, multiply them.
   (b) If they are **above and below** each other, divide the top quantity by the bottom quantity.

This trick accomplishes the same thing as rearranging the equation, but it is important to understand how the triangle is working. You are given that the relationship is $v = d \div t$. Say that you want to solve for the displacement. The goal is to have d by itself. To accomplish this, multiply both sides of the equation by t. The result is $d = v \times t$.

Try it out:

Explain how to rearrange $v = d \div t$ to find t.

Multiply both sides by t. Then, divide both sides by v.

Examples

The easiest way for you to learn how to solve kinematics problems is by looking at examples.

Example 1

Harry is chasing the snitch. The snitch is not moving. After flying for 6 seconds, he catches the snitch. He covered a total of 42 metres in his pursuit. What was his (average) velocity?

Given:

$t = 6$ s

$d = 42$ m

Required:

$v = ?$

Application:

$v = d \div t$

Solution:

The equation is already in the form $v = ?$ so we can simply plug the numbers in.

$v = 42 \div 6 = 7$.

Statement:

$\therefore$ Harry's average velocity was 7 m/s.

Example 2

Crookshanks is chasing Scabbers. Scabbers becomes stuck to the floor 2 metres in front of Crookshanks and is unable to move. If Crookshanks is running at 40 cm/s, how long does it take him to catch Scabbers?

Given:

$v = 40$ cm/s $= 0.4$ m/s

$d = 2$ m

Required:

$t = ?$

Application:

$v = d \div t$

Solution:

Instead of rearranging the equation or using the triangle right away, let's stick what we know in the equation. This gives us $0.4 = 2 \div t$. Now, multiply both sides by t and then divide by 0.4. This gives us $t = 5$.

Statement:

$\therefore$ Crookshanks catches Scabbers after 5 seconds.

Example 3

Crookshanks is chasing Scabbers again. If Scabbers starts 2 metres in front of Crookshanks and runs at 30 cm/s while Crookshanks runs at 40 cm/s, how long does it take Crookshanks to catch Scabbers?

Given:

$v_C = 40$ cm/s $= 0.4$ m/s

$v_S = 30$ cm/s $= 0.3$ m/s

Required:

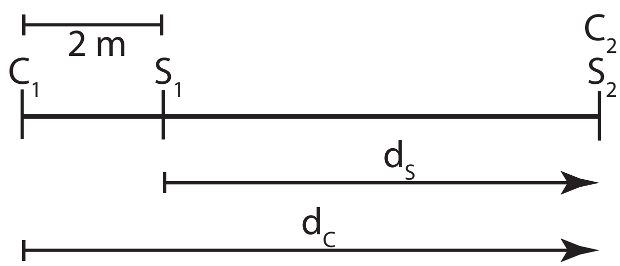
$t = ?$

Application:

$v = d \div t$

Solution:

This problem is more challenging because both objects are moving. We must find the relationships between the three quantities of each object. While there are different ways to solve this problem, it is important to start with a picture.



In the picture, C_1 is where Crookshanks starts, and S_1 is where Scabbers starts. When Crookshanks catches Scabbers, they will be at the same location. Letting d_C be the displacement of Crookshanks and d_S be the displacement of Scabbers, we see from the picture that $d_C = d_S + 2$. It is important to realize that the length of time they have both run is the same. Thus, $t_C = t_S = t$.

One way to go from here is to write the equation in terms of d for both objects. So, $d_C = v_C \times t$ and $d_S = v_S \times t$. But we can replace d_C with $d_S + 2$. This makes the first equation $d_S + 2 = v_C \times t$. We can rearrange this to read $d_S = (v_C \times t) - 2$.

Now we have two equations that both equal d_S, so let's set them equal to each other. This gives us $v_S \times t = (v_C \times t) - 2$. Plugging in the known values for v_C and v_S we get $0.3 \times t = (0.4 \times t) - 2$. Rearranging and solving, we get $0.1 \times t = 2$, giving us $t = 20$.

Statement:

$\therefore$ Crookshanks catches Scabbers after 20 seconds.

Example 4

Fred stands 5 metres in front of George. If Fred starts walking at a speed of 2 m/s at the same time as George starts walking at a speed of 1 m/s, how long does it take them to walk a combined total of 9 m?

Given:

$v_F = 2$ m/s; $v_G = 1$ m/s

Required:

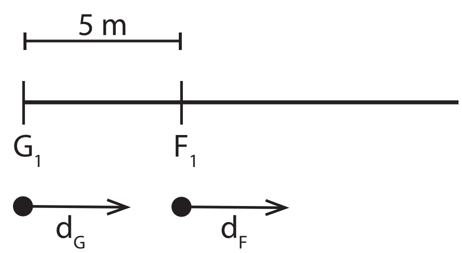
$t = ?$

Application:

$v = d \div t$

Solution:

As in the previous example, we want to find the relationships between the displacements and times of the two objects. It is helpful to start with a picture.



As in the previous example, the amount of time both objects are moving is the same, so $t_G = t_F = t$. But how do we relate the displacements? Well, Fred & George stop moving when they have walked a combined total of 9 metres. This gives us the relationship $d_G + d_F = 9$. Notice that the initial space between them does not matter for this question.

One way to go from here is to write the equation in terms of t for both objects. So, $t = d_F \div v_F$ and $t = d_G \div v_G$. Since these are both equal to t, we can set them equal to each other. We can also plug in the values of v_G and v_F.

$d_F \div 2 = d_G \div 1$

$d_F = 2 \times d_G$

We can now replace d_F with $9 - d_G$.

$9 - d_G = 2 \times d_G$

$9 = 3 \times d_G$

$d_G = 3$

We've found that George walks 3 metres. But we want to know how long he has been walking. So, we can plug this value into the equation for George's time. $t = d_G \div v_G = 3 \div 1 = 3$.

Statement:

$\therefore$ Fred and George take 3 seconds to walk a combined total of 9 metres.

Wrap-Up

Today, you took a look at a real-world application of variables. Variables are constantly used in higher mathematics to solve problems. You will see as you progress through your studies that it is important to have a solid foundation in terms of solving equations with unknowns.

Problem Set

Complete the following 10 problems. You may use a calculator.

1. Neville thinks practice is important. Redo the warm-up. See if you can answer more questions than before!

2. Argus can run at a velocity of 7 200 000 mm/hr. What is this in standard units?
   2 m/s

$$\begin{aligned}\frac{7200000\text{ mm}}{1\text{ hr}} &= \frac{7200000\ \cancel{\text{mm}}}{1\text{ hr}} \times \frac{1\text{ m}}{1000\ \cancel{\text{mm}}}\\ &= \frac{7200\text{ m}}{1\ \cancel{\text{hr}}} \times \frac{1\ \cancel{\text{hr}}}{3600\text{ sec}}\\ &= \frac{7200\text{ m}}{3600\text{ sec}}\\ &= 2 \times \frac{1\text{ m}}{1\text{ sec}}\end{aligned}$$

3. Igor bats a bludger towards Viktor. Viktor is not moving. If the bludger travels at a constant velocity of 3 m/s and hits Viktor 2 seconds later, how far away was Viktor from Igor? $d = v \times t \implies d = 3 \times 2 = 6$ metres

4. Igor bats another bludger at Viktor. Viktor is 5 metres away from Igor. This time, Viktor is moving away at a velocity of 4 m/s. The bludger moves at a velocity of 3 m/s. Will the bludger hit Viktor? If so, after how long? How far has the bludger travelled?
   The bludger won't hit Viktor as it is moving slower than he is, so it will never catch up!

5. Igor bats one last bludger at Viktor. Viktor is 5 metres away from Igor. This time, Viktor is moving away at a velocity of 1 m/s. The bludger moves at a velocity of 3 m/s. Will the bludger hit Viktor? If so, after how long? How far has the bludger travelled?
   The bludger will hit Viktor. Draw a picture of the situation, then solve the problem.
   $v_V = 1, v_B = 3$
   $d_B = d_V + 5$
   $t_B = t_V = t$
   Setting the equations for t equal to each other:
   $d_B \div v_B = d_V \div v_V$
   Substituting:
   $(d_V + 5) \div 3 = d_V \div 1$
   $(d_V + 5) \div 3 \times 3 = d_V \times 3$
   $d_V + 5 = 3d_V$
   $d_V + 5 - d_V = 3d_V - d_V$
   $5 = 2d_V$
   $d_V = 2.5$
   $\therefore$ The bludger travels 2.5+5 = 7.5 metres to hit Viktor.
   It takes $t = d_V \div v_V = 2.5 \div 1 = 2.5$ seconds for the bludger to hit Viktor.

6. Ron is flying his Ford Anglia over London. The speed limit for flying cars is 75 km/h. Ron is able to fly from Hyde Park to Regent's Park in 1.5 minutes. The distance between the two parks is 2 kilometres. Is Ron speeding? If so, by how much?
   Start by converting to standard units. $t = 1.5 \times 60 = 90$ s and $d = 2 \times 1000 = 2000$ m.
   $v = d \div t = 2000 \div 90 = 22.\overline{2}$ m/s
   Convert this to km/hr: $22.\overline{2} \times 3.6 = 80$ km/hr
   $\therefore$ Ron is speeding by 5 km/hr.

7. Hermione and Ginny are running through a wall (a magical doorway) and will exit on Platform $9\frac{3}{4}$. They both start 10 metres away from the wall. Hermione runs at 2.5 m/s and Ginny runs at 2 m/s.

   (a) How long is it before Hermione enters the wall? $t = d \div v = 10 \div 2.5 = 4$ seconds

   (b) If Hermione exits the wall just as Ginny enters it, how thick is the wall?
   We need to find how long it is before Ginny enters the wall.
   $t = d \div v = 10 \div 2 = 5$. So Hermione is travelling through the wall for 5-4 = 1.
   The wall's thickness is the distance Hermione runs through the wall.
   $d = v \times t = 2.5 \times 1 = 2.5$ metres. $\therefore$ The wall is 2.5 m thick.

8. Mundungus robs Gringotts bank and flees at a constant velocity. 10 minutes later, the security goblins leave the bank to pursue him, moving at a velocity 5 m/s greater than Mundungus's velocity. The goblins catch up to Mundungus 20 minutes later. How fast was Mundungus going? How fast were the goblins going?

   Mundungus is caught 30 minutes after he flees. So $t_M = 1800$ s. The goblins catch Mundungus after 20 minutes. So $t_G = 1200$ s. When the goblins catch Mundungus, they of them will have travelled the same distance from the bank. $\therefore d_M = d_G = d$. The relationships between the velocities is $v_G = v_M + 5$. We can put both equations in terms of d and set them equal to each other.

   $v_G \times t_G = v_M \times t_M$

   $(v_M + 5) \times 1200 = v_M \times 1800$

   $(v_M + 5) \times 1200 \div 1200 = v_M \times 1800 \div 1200$

   $(v_M + 5) = v_M \times 1.5$

   $v_M + 5 = 1.5v_M$

   $v_M + 5 - v_M = 1.5v_M - v_M$

   $5 = 0.5v_M$

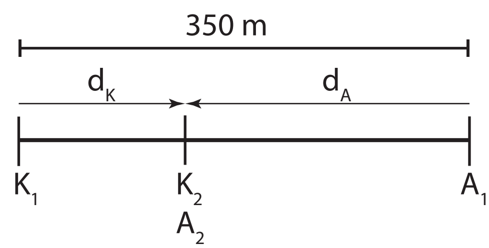
   $10 = v_M$

   $\therefore$ Mundungus was going 10 m/s and the goblins were going 10+5 = 15 m/s.

9. Katie Bell and Angelina Johnson are on a collision course. Initially, they are 350 metres apart. Katie moves at 21 km/h and Angelina moves at 35 km/h. If they both started moving at the same time, how long is it before they collide? What displacement did Angelina travel?

Converting to standard units, $v_K = 5.8\overline{3}$ and $v_A = 9.7\overline{2}$.

Draw a picture of the situation:



We find that the relationships are $t_K = t_A = t$ and $d_A + d_K = 350$

$d_K = v_K \times t$

Substituting:

$350 - d_A = v_K \times t$

$d_A = 350 - v_K t$

Substituting:

$v_A t = 350 - v_K t$

$9.7\overline{2}t + 5.8\overline{3}t = 350$

$15.\overline{5}t = 350$

$t = 22.5$

$\therefore$ It takes 22.5 seconds for them to collide.

Substituting into $d_A = v_A \times t$, we find $d_A = 9.7\overline{2} \times 22.5 = 218.75$

$\therefore$ Angelina travelled 218.75 metres.

10. Two trains, the Hogwarts Express and the Waterloo Direct, are initially 300 km apart. There is an owl on the Hogwarts Express. At the same time, both trains start moving towards each other, each travelling at 45 km/h. While this is happening, the owl is flying back and forth between the trains, turning around every time it reaches one. If the owl is flying at 36 km/h, how far does it travel before the two trains collide?

    The trick to this question is HOW to solve it. The actual math is the same as in question 9, but a bit easier. We are looking for the distance the owl flies. The fact that it repeatedly turns around doesn't matter. We are given the owl's velocity. We want the owl's distance travelled. This means we need to know the length of time the owl was flying. This is the time it takes for the trains to collide.

    You could use the same approach as question 9 to find the length of time, but because these trains are travelling at the same speed, you can use logic. Since they are travelling at the same speed, they will collide exactly midway. So you just need to know the length of time it takes one train to travel 150 km. Plugging into $t = d \div v$ we find that $t = 3.\overline{3}$ hours.

    Imagine the owl does not repeatedly turn around, but flies continuously in one direction. Then, $d = v \times t$ gives us 120 000 metres, or 120 kilometres.
    $\therefore$ the owl flies 120 km before the trains collide.