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Grade 7/8 Math Circles

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Arithmetic Aerobics

Mental Math is Good For You!

You've probably heard that it's important to keep your body healthy. Well, the same is true for your brain! As mathematicians, it's important to ***practice*** our mental math skills in order to keep our brains sharp.

Today, we're going to start with some neat tricks to help you with the fundamental operations of arithmetic. Then, we're going to look at some real world applications involving percentages and conversions.

Before we begin, it's important to understand what we're dealing with:
Arithmetic is the oldest branch of mathematics. Simply put, it is the study of numbers and the operations between them.

Warm-Up

Try to answer the following 14 questions in 6 minutes without a calculator. Don't worry if you can't answer them all. You'll be an expert by the end of this lesson!

1. $\begin{array}{r} 7 \\ \times 9 \end{array}$

2. $\begin{array}{r} 17 \\ \times 11 \end{array}$

3. $\begin{array}{r} 52 \\ \times 37 \end{array}$

4. $\begin{array}{r} 34 \\ \times 45 \end{array}$

5. $\begin{array}{r} 323 \\ \times 19 \end{array}$

6. Is 45 319 divisible by 3?

7. Is 725 128 divisible by 4?

8. Is 34 272 divisible by 6?

9. Evaluate 13^2.

10. Evaluate 105^2.

11. Hermione wants to buy chocolate that is \$3.30 plus 10% tax. How much money must she have to purchase this chocolate?

12. Billius wants to buy a toy that is \$5.00 plus 15% tax. How much money must he have to purchase this toy?

13. Ryan says he ran 10 kilometres today. How many metres did he run?

14. Sachin has a field with area 10 000 cm^2. What is the area of the field in m^2?

Single Digit Multiplication by 9

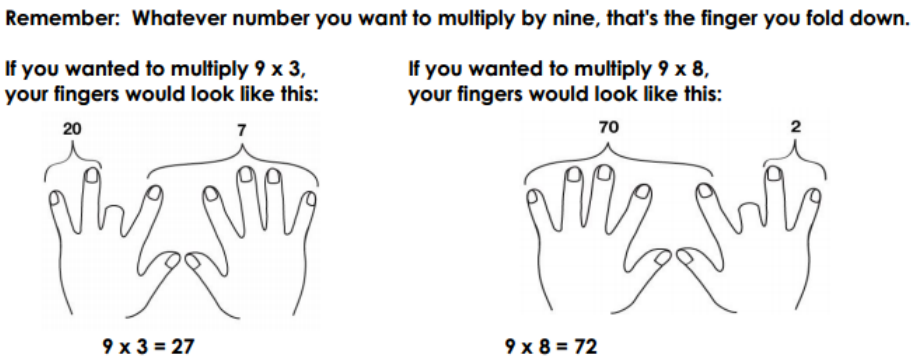
Warm-Up (WU) 1 tested your ability to multiply by 9 quickly. There is a very simple trick for quickly multiplying a number from 1-10 by 9:

1. Put both your hands in front of you.

2. When you are asked to multiply a number from 1-10 by 9, you put that number finger (let's say thumbs are fingers too) from the left down.

3. The number of fingers to the left of the one you put down is the tens digit of the answer.

4. The number of fingers to the right of the one you put down is the ones digit of the answer.



Picture from: http://www.superteacherworksheets.com/multiplication/nines-trick.pdf

Try it out:

$6 \times 9 =$ $9 \times 4 =$

Multiplication by 11

WU 2 tested your ability to multiply by 11 quickly. To multiply a single digit by 11, you simply write the digit twice. It gets a bit more interesting when we want to multiply a multi-digit number by 11:

1. Given a number, first write down its rightmost digit.
2. Then add its rightmost digit with the digit on the left of it, put down the *ones* digit of the result to the left of the digit you previously put down.
3. Remember the carry-over digit, if there is any.
4. Repeat step 2 for the second rightmost digit, adding in the previous carry-over, if there is any, and so on, until you are at the leftmost digit (and so there are no more numbers to its left).
5. Add the carry-over from previous addition of the second leftmost digit with the leftmost digit, if there is any, to this number, and put it down on the leftmost side of your result.

Try it out:

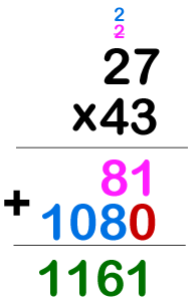
$6 \times 11 =$ $\quad$ $11 \times 214\,353 =$ $\quad$ $13\,856 \times 11 =$

Two Digit Multiplication

WU 3 and 4 tested your ability to multiply two-digit numbers together. This is a vital skill for any budding mathematician. You will find that the more you *practice* this type of multiplication, the better (and faster) you will get! Here's how to approach it in stacked form:

1. Multiply the ones digit of the bottom number by the ones digit of the top number.
2. If the answer is one digit, write it down. If it is two digits, write down the ones digit and carry over the tens digit.
3. Multiply the ones digit of the bottom number by the tens digit of the top number. Make sure to add the carry-over, if there is one. Write the result down.
4. Put a **placeholder zero** as the ones digit on the next line.
5. Repeat steps 1 & 2 for the tens digit of the bottom number.
6. Add the two lines together.

Take a look at this example:



Try it out:

1. $\begin{array}{r} 54 \\ \times 90 \\ \hline \end{array}$
2. $\begin{array}{r} 73 \\ \times 11 \\ \hline \end{array}$
3. $\begin{array}{r} 85 \\ \times 37 \\ \hline \end{array}$
4. $\begin{array}{r} 34 \\ \times 17 \\ \hline \end{array}$

The above technique can be extended to the multiplication of any two multi-digit numbers!

A General Multiplication Trick

WU 5 tested your ability to multiply large numbers using simpler multiplication. Of course, you could use the previous technique to solve WU 5. However, the following trick is one of the best ways to become faster at multiplication problems. Remember, mathematics is all about using what you know to build something new.

In warm-up 5, you were asked to compute 323×19. Well, recall that multiplication is just repeated addition. Then, why don't we find 323×20 (a much easier quantity to compute) and subtract 323 once from that? We can say that $323 \times 20 = 323 \times 2 \times 10 = 646 \times 10 = 6460$. Now, $6460 - 323 = 6137$. By using the previous technique, you can confirm that $323 \times 19 = 6137$.

Try it out:

$17 \times 49 =$ $5482 \times 99 =$ $21 \times 213 =$

Divisibility Rules

WU 6, 7, and 8 tested your knowledge of divisibility rules. You could answer all of these questions using long division, but mathematicians are lazy and long division would take too long. Instead, we have discovered some neat tricks to speed up the process. Some of the simplest and most common can be found in the table below.

Divisible by:	**Test:**
2	The last digit in the number is even
3	The sum of all the digits is divisible by 3
4	The last two digits are divisible by 4
5	The number ends in 0 or 5
6	The number is divisible by 2 *and* 3
8	The last three digits are divisible by 8 Note: Can't use this rule for numbers less than 1000
9	The sum of the digits is divisible by 9
10	The number ends in 0
11	The *alternating* sum of the digits is divisible by 11 Notes: Start with subtraction. 0 is divisible by 11
12	The number is divisible by 3 *and* 4

Try it out:

1. Is 34 627 888 divisible by 8?
2. Is 457 925 divisible by 9?
3. Is 6756 divisible by 12?

Squaring Numbers

WU 9 and 10 tested your ability to square numbers. Recall that exponents signify repeated multiplication. Thus, $7^4 = 7 \times 7 \times 7 \times 7$. ***As a young mathematician, it is recommended that you memorize the squares of numbers from 0 to 20.*** Just like the multiplication table, squares are an important building block in your knowledge of mathematics.

There is a cool trick for squaring numbers greater than 99 that end in 5:

- Notice that numbers greater than 99 that end in 5 can be written in the form $(10\times n)+5$.
- To find the square, first write down the product $n \times (n+1)$. Then, write a 25 to the right of the product you found.

For example, $115 = (10\times 11)+5$, so $n = 11$. Then $11 \times 12 = 132$. So $115^2 = 13\,225$.

Try it out:

$295^2 =$ $1015^2 =$

Taxes

WU 11 and 12 tested your ability to use percentages in a real life application. When using percentages, it is important to remember their relationship with decimals and fractions. For example, $20\% = 0.2 = \frac{20}{100} = \frac{1}{5}$.

When you buy something from a store in Ontario, you must pay the price on the tag, *plus* tax. This is an extra amount stated by the government in terms of a percentage. Currently, we have a Harmonized Sales Tax (HST) in Ontario of 13% on most items. Certain items, such as books, only have a 5% tax. The moral of this story is that you should spend more time reading!

Let's do an example. Fleur wants to buy some flowers. The price tag says \$5.00. The tax where Fleur lives is 10%. How much will she have to pay?

To solve this problem, recall that $10\% = 0.1 = \frac{1}{10}$. Multiplying by one tenth is the same as dividing by 10. You should understand that multiplying or dividing by 10 results in simply moving the decimal place. This is thanks to the fact that we work in the *decimal* system! You may have also heard it being called "base 10" in your math class. So, $5.00 \times \frac{1}{10} = 0.50$. But we are not done yet! We have only found that the tax itself is \$0.50. We must add this to the original price to find the total cost. Thus, $\$5.00 + \$0.50 = \$5.50$. So, Fleur must pay \$5.50.

This problem was pretty easy since the tax was 10%. What if the tax was 15% or 13% (like in Ontario)? You could of course do the multiplication using the decimal form, but multiplication with 0.15 or 0.13 is much harder to calculate mentally. An approach you could use is *estimation*.

You will notice that 0.15 is exactly in the middle of 0.1 and 0.2. In fact, it is the *average* or *mean* of 0.1 and 0.2. So to calculate a 15% tax, we could mentally calculate the tax if it were 10% and 20% (use the fact that $0.2 = 0.1 \times 2$) and then find the average of the two values. This would give us the value of a 15% tax exactly. To approximate a 13% tax mentally, we could find the value of the tax if it were 15% (using the above technique) and 10%. Taking the average of the two values would give us a reasonable approximation (it would actually give us the value for a 12.5% tax).

Try it out:

Tyrion wants to purchase a soccer ball. The price tag says the ball costs \$15.60. The tax where Tyrion lives is 13%. Using the above technique, estimate how much Tyrion will have to pay (without a calculator).

Metric Conversions

WU 13 and 14 tested your ability to convert quantities within the *metric* system. Conversions in the metric system are quite simple as it is a decimal system. Some common conversions can be found below. You should understand how to do conversions that are not explicitly given below.

1 kilometre (km)	1000 metres (m)
1 metre	100 centimetres (cm)
1 centimetre	10 millimetres (mm)
1 centimetre	0.01 metres
1 kilogram (kg)	1000 grams (g)
1 gram	1000 milligrams (mg)
1 gram	0.001 kilograms
1 litre (L)	1000 millilitres (mL)

It is also important for you to be able to convert between units of volume and area. To do this, we extend the conversions that we already know. We will use the conversion factors above to build new conversion factors.

Let's do an example. Mary had a little lamb. She decided to build a cage with volume $8\,000\,000$ cm^3 for her lamb. However, the cage-builders only understand units of m^3 and mm^3. What is the volume of the cage in m^3 and mm^3?

We know that 1 cm = 0.01 m. You should understand how to find this conversion using the table. But we need to find m^3. Recall that this is the same as m$\times$m$\times$m. Thus our conversion factor is $0.01 \times 0.01 \times 0.01 = 0.000001$. So, 1 cm^3 = 0.000001 m^3. Then, the volume of the cage is $8\,000\,000 \times 0.000001 = 8$ m^3. Similarly, we find that the volume of the cage is $8\,000\,000\,000$ mm^3.

Try it out:

1. Convert 273 mL to L.
2. Convert 63 km to mm.
3. Convert 2 km^2 to cm^2.

Wrap-Up

Every bit of math you did today was done without a calculator! You've completed an aerobics class for your brain. As you advance in your studies, you will find that practicing the mental math techniques you've learned today will help you become quicker at calculations and assist you in becoming a stronger mathematician.

Problem Set

Complete all 22 problems without a calculator.

1. Redo the warm-up. See if you are faster than before!
2. $8 \times 9 =$
3. $9 \times 6 =$
4. $11 \times 3 =$
5. $547\,987 \times 11 =$

6. $\begin{array}{r} 18 \\ \times 65 \\ \hline \end{array}$

7. $\begin{array}{r} 76 \\ \times 72 \\ \hline \end{array}$

8. $\begin{array}{r} 23 \\ \times 98 \\ \hline \end{array}$

9. $\begin{array}{r} 492 \\ \times 533 \\ \hline \end{array}$

10. $47 \times 27 =$
11. $368 \times 58 =$
12. $98 \times 417 =$
13. Is $47\,854\,389$ divisible by 9?
14. Is $47\,854\,389$ divisible by 3?
15. Is $47\,424$ divisible by 4?

16. $15^2 =$

17. $12^2 =$

18. $915^2 =$

19. $4005^2 =$

20. Ronald wants to buy a monkey. The price tag says \$475.20. The tax rate where Ronald lives is 17%. **Approximately** how much does Ronald have to pay for the monkey?

21. Ronald decides to buy a cage for his monkey. He decides that the cage must have a volume of 0.000000017 km^3. The cage-builders only understand mm^3. How large is the cage in mm^3?

22. Ronald needs to provide a water tank for his monkey. Tanks are labelled with the mass of water they can hold in kg. Ronald needs a water tank that can hold 72 L of water. How many kg must the water tank hold? Use the fact that 1 mg of water is equivalent to 0.000001 L of water.