Math Circles - April 1st 2015 - Infinite Series
Instructor: Jordan Hamilton
SOLUTIONS

1. Determine if the following series converge or diverge, if the series converges find its sum.

(a) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{13}{12}\right)^n$.

Solution: This series diverges since $r = \frac{13}{12} \geq 1$.

(b) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2}{7}\right)^{n+1}$

Solution: This series converges, $r = \frac{2}{7} < 1$, and we get

$$\sum_{n=0}^{\infty}\left(\frac{2}{7}\right)^{n+1} = \sum_{n=0}^{\infty}\frac{2}{7}\cdot\left(\frac{2}{7}\right)^n = \frac{2}{7}\cdot\frac{1}{1-\frac{2}{7}} = \frac{2}{5}.$$

(c) $\displaystyle\sum_{n=0}^{\infty}2^{3n}\cdot 5^{2-n}$.

Solution: Let's simplify first:

$$\sum_{n=0}^{\infty}2^{3n}\cdot 5^{2-n} = \sum_{n=0}^{\infty}2^{3n}\cdot 5^2\cdot 5^{-n} = \sum_{n=0}^{\infty}25\cdot\frac{8^n}{5^n} = \sum_{n=0}^{\infty}25\cdot\left(\frac{8}{5}\right)^n$$

and so this series diverges since $r = \frac{8}{5} \geq 1$.

(d) $\displaystyle\sum_{n=0}^{\infty}\frac{2^{n-1}}{3^{n+1}}$.

Solution:

$$\sum_{n=0}^{\infty}\frac{2^{n-1}}{3^{n+1}} = \sum_{n=0}^{\infty}\frac{1}{6}\cdot\frac{2^n}{3^n} = \sum_{n=0}^{\infty}\frac{1}{6}\cdot\left(\frac{2}{3}\right)^n = \frac{1}{6}\cdot\frac{1}{1-\frac{2}{3}} = \frac{1}{2}.$$

In this case $r = \frac{2}{3} < 1$.

(e) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{4}{9}\right)^n$

Solution: Clearly the series converges since $r = \frac{4}{9} < 1$, but the index of this series doesn't start at the usual $n = 0$. If n were zero then the first term would be 1, so let's add and subtract that:

$$\sum_{n=1}^{\infty}\left(\frac{4}{9}\right)^n = -1 + 1 + \sum_{n=1}^{\infty}\left(\frac{4}{9}\right)^n = -1 + \sum_{n=0}^{\infty}\left(\frac{4}{9}\right)^n = -1 + \frac{1}{1-\frac{4}{9}} = -1 + \frac{9}{5} = \frac{4}{5}.$$

2. Write the following infinite decimals as a single fraction

   (a) $0.1919191919\ldots = 0.\overline{19}$.

   Solution:

$$0.\overline{19} = \frac{19}{100} + \frac{19}{100^2} + \cdots = \frac{19}{100}\left(1 + \frac{1}{100} + \frac{1}{100^2} + \cdots\right) = \frac{19}{100}\sum_{n=0}^{\infty}\left(\frac{1}{100}\right)^n$$

   which is a convergent series (since $r = \frac{1}{100} < 1$) that sums up to

$$\frac{19}{100} \cdot \frac{1}{1 - \frac{1}{100}} = \frac{19}{99}.$$

   (b) $1.123412341234\ldots = 1.\overline{1234}$.

   Solution:

$$1.\overline{1234} = 1 + \frac{1234}{10000} + \frac{1234}{10000^2} + \cdots = 1 + \frac{1234}{10000}\left(1 + \frac{1}{10000} + \frac{1}{10000^2} + \cdots\right)$$

$$= 1 + \frac{1234}{10000}\sum_{n=0}^{\infty}\left(\frac{1}{10000}\right)^n.$$

   Again, this series converges to

$$1 + \frac{1234}{10000} \cdot \frac{1}{1 - \frac{1}{10000}} = 1 + \frac{1234}{9999} = \frac{11233}{9999}.$$

3. Suppose you tweak the defense shields around the planet so that they now reflect $\frac{1}{3}$ of the beam, absorb $\frac{5}{9}$ of the beam and transmit $\frac{1}{9}$ of the beam. What fraction of the starting intensity (I) gets through now?

   Solution: The set up and picture are similar to the one in the notes. But in this case the total beam intensity that gets through is

$$\frac{I}{81} + \frac{I}{9(81)} + \frac{I}{9^2(81)} + \cdots = \frac{I}{81}\sum_{n=0}^{\infty}\left(\frac{1}{9}\right)^n = \frac{I}{81} \cdot \frac{1}{1 - \frac{1}{9}} = \frac{I}{81} \cdot \frac{9}{8} = \frac{I}{72}.$$

   That's better than before!

4. Prove that $\displaystyle\sum_{n=1}^{\infty}\left[(-1)^n\left(\frac{2}{n}\right)\cos^2\left(\frac{n\pi}{2}\right)\right]$ diverges.

   Solution: This series looks ridiculous! But let's examine each part. The $(-1)^n$ term alternates between -1 if n is odd and 1 if n is even. The $\frac{2}{n}$ term looks like twice the harmonic series.

Finally, what does $\cos^2\left(\frac{n\pi}{2}\right)$ look like? If n is even, say $n = 2k$, then we get $\cos^2(k\pi) = 1$. On the other hand, if n is odd then $\cos^2\left(\frac{n\pi}{2}\right) = 0$. Therefore, the overall term is zero if n is odd! So let's only consider the even n, let $n = 2k$ and we get

$$\sum_{n=1}^{\infty}\left[(-1)^n\left(\frac{2}{n}\right)\cos^2\left(\frac{n\pi}{2}\right)\right] = \sum_{k=1}^{\infty}\left[(-1)^{2k}\left(\frac{2}{2k}\right)\cos^2(k\pi)\right] = \sum_{k=1}^{\infty}\frac{1}{k}.$$

But that's just the harmonic series! We've already proven that it diverges in the lecture.

5. Find a simple expression for the nth partial sum of the series and use it to find the sum of the series:

$$\sum_{n=0}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Such series are called *telescoping series* and are another example of a series in which we can actually find the sum.

Solution: Let's start looking at the partial sums.

$$S_1 = \frac{1}{1} - \frac{1}{2},$$

$$S_2 = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{1}{1} - \frac{1}{3},$$

$$S_2 = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = \frac{1}{1} - \frac{1}{4},$$

and so on. Now we can see the pattern:

$$S_n = 1 - \frac{1}{n+1}.$$

What happens to S_n as $n \to \infty$? Clearly $\frac{1}{n+1} \to 0$ as $n \to \infty$, so $S_n \to 1$. This series adds up to 1!

6. What's wrong with the following argument?

Let $x = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ and $y = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots$. Then clearly $2y = x + y$ which means $x = y$. But notice that

$$x - y = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots > 0$$

since it's a sum of positive numbers. Therefore $x > y$. So we have shown that $x = y$ and $x > y$.

Solution: The problem is actually right at the start. Both x and y are infinite! Each is some part of the harmonic series (x has the odd terms and y has the even terms). Therefore it doesn't make sense to talk about their values nor do arithmetic with them!