Math Circles
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Paradoxes
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SOLUTIONS

Introduction Paradoxes:

1. **The Potato Paradox**:
   Suppose you have 100 pounds of potatoes, and they're special potatoes in that they are 99% water. Let's say you let them dehydrate for awhile, until they are only 98% water. How much do they weigh now? The surprising answer is: 50 pounds!

   (a) Is the statement true? If so, can you prove it? If not, why not?
   Solution: It's true, surprisingly. Here are two ways to see it.

   Method 1: Since the potatoes are 99% water, they contains $0.99*100 = 99$ pounds of water. That leaves 1 pound of solids. If we decrease them to 98% water then that means the 1 pound of solids actually makes up 2% of the whole weight. But 1 is 2% of 50, so the new total weight is 50 pounds.

   Method 2: Let's prove it more rigorously. Suppose x is the weight of water lost. So the new dehydrated potatoes weigh $100 - x$ pounds. At the start, there is $0.99 * 100 = 99$ pounds of water in the potatoes. At the end, there would be $0.98 * (100 - x) = 98 - 0.98x$ pounds of water. The difference is the amount of water lost, that is x. So we get the equation

   $$99 - (98 - 0.98x) = x.$$

   Let's solve the equation for x:

   $$1 + 0.98x = x$$

   so

   $$1 = 0.02x$$

   which means

   $$x = \frac{1}{0.02} = 50.$$

   So the potatoes lose 50 pounds of water.

   (b) What kind of paradox is this?
   Solution: A truthful paradox. It sure seems like too much water lost, but the proofs above are correct!

2. Let's think about a Disney cartoon for a second: Pinocchio! Pinocchio is a wooden doll whose nose will get longer if (and only if) he tells a lie.

   (a) What happens if Pinocchio says 'my nose will get longer now'?
   Solution: It wont grow, see below.

   (b) What kind of paradox is this in part (a)?
   Solution: An antinomy, the statement has no truth value. Note that this means his nose wont grow as what he said is neither true nor false (so it can't be a lie).

3. Consider the following proof: We know that $\cos^2(x) + \sin^2(x) = 1$ for any $x \in \mathbb{R}$. So we get:

$$\begin{aligned} \cos^2(x) &= 1 - \sin^2(x) \\ \cos(x) &= \left(1 - \sin^2(x)\right)^{\frac{1}{2}} \\ 1 + \cos(x) &= 1 + \left(1 - \sin^2(x)\right)^{\frac{1}{2}} \end{aligned}$$

   Now let's substitute $x = \pi$ to get

$$1 - 1 = 1 + (1 - 0)^{\frac{1}{2}}$$

   or

$$0 = 2.$$

   (a) This can't be true, can it? What's wrong with the 'proof'?
   Solution: When we took the square root in the second step we didn't include the $\pm$. It's true with a minus sign (as it says it's $+$ OR $-$).

   (b) What kind of paradox is this?
   Solution: It's a fallacy.

4. **All horses are the same colour**, did you know that? Let's prove it with mathematical induction!

   Let's let P_n be the statement 'every collection of n-many horses are all the same colour as each other'. Let's show that P_n is true for all n.

   Base Case: $n = 1$. Obviously every collection of a single horse are all the same colour as each other, so the $n = 1$ case is done.

   Inductive Step: Suppose that every collection containing exactly k-many horses are all the same colour as each other. We want to show that every collection of $k+1$ many horses all have the same colour.

So take any collection of $k+1$-many horses. Now remove one horse at random. Then we're left with only k-many horses which, by assumption, all have the same colour as each other.

Ok, put that horse we took away back in, and take a DIFFERENT horse out. Now, again, there are k-many horses so they are all the same color. So put that horse back, and it will be the same colour as the rest.

So all $k+1$-many horses have the same colour, which finishes our mathematical induction proof.

Questions:

(a) Surely there's something wrong with the above argument, what is it?
Solution: The trick is hidden pretty good in this one. The problem is with $k = 2$! If there are ONLY two horses, and we take one away then sure, every horse has the same colour. Then we put that one back and take the other one away and still every horse has the same colour. BUT we didn't say those two horses had the *same* colour!

When we presented this argument we implicitly assumed that taking away one horse left *more than one* horse behind, which isn't the case if there are only two horses.

(b) What kind of paradox is this?
Solution: It's a fallacy.

5. **Devil's Dice**: Consider three dice A, B, and C.

- Die A has sides $2, 2, 4, 4, 9, 9$;
- Die B has sides $1, 1, 6, 6, 8, 8$;
- Die C has sides $3, 3, 5, 5, 7, 7$.

(a) What is the probability that die A rolls higher than die B?
Solution: There are 36 possible combinations you can get when rolling two six-sided dice. Let's figure out what combinations have A winning.

First, if die A rolls a 9 then no matter what B rolls it will be lower. So that's $6 + 6 = 12$ ways A can beat B.

Next, if A rolls a 4 then it only wins if B rolls a 1. There are two 1's on B, so that's $2 + 2 = 4$ ways A could win in this case.

Finally, if A rolls a 2, then again it only wins if B rolls a 1, so another $2 + 2 = 4$ ways A could win.

This gives $12 + 4 + 4 = 20$ out of the possible 36 combinations as victories for A. This is a probability of

$$\frac{20}{36} = \frac{5}{9}.$$

This means A wins more than half the time.

(b) What is the probability that die B rolls higher than die C?
Solution: A similar argument works here, we get the probability to be $\frac{5}{9}$. So B beats C more than half the time too!

(c) What is the probability that die C rolls higher than die A?
Solution: Similarly, the probability is $\frac{5}{9}$. So C beats A more than half the time! Crazy!

(d) What kind of paradox is this?
Solution: It's a truthful paradox! We've just shown that there are 3 dice A, B, C such that A beats B more than half the time, B beats C more than half the time, and C beats A more than half the time! It doesn't seem like such dice could exist but here they are.

6. **Hilbert's Grand Hotel**. Consider a hotel with an infinite number of rooms, labeled $1, 2, 3, \ldots$. Also suppose every room is occupied. You'd think that no new guests could fit, right? Well that's only true of our boring finite hotels.

(a) How could you fit another guest into the hotel by shifting people to different rooms?
Solution: Ask the person in room 1 to move to room 2, the person in room 2 to move to room 3, and so on. In general, the person in room k moves to room $k + 1$. Then room 1 will be empty for the new guest.

(b) What about finitely many (say N) new guests?
Solution: Similarly to part (a), but move the guest in room k to room $k + N$. This will leave the first N rooms empty for the new guests.

(c) What about *infinitely* many new guests?
Solution: This is trickier, we want to clear out an infinite number of rooms. Let's use the even and odd numbers to our advantage! We'll clear out all the odd numbered rooms.
Ask the person in room 1 to move to room 2, the person in room 2 to move to room 4, the person in room 3 to move to room 6 and so on. In general, the person in room k moves to room $2k$. Then only even-numbered rooms will have guests, and we can put our infinitely many new guests into the odd numbered rooms!

(d) What kind of paradox is this?
Solution: This is a truthful paradox. Stuff like this would be no problem if we could build a hotel with infinitely many rooms!