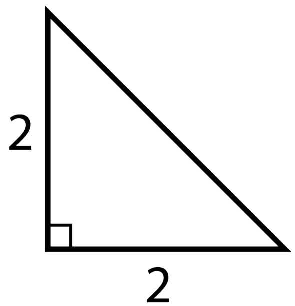
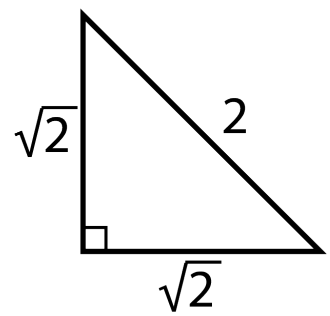
Intermediate Math Circles
November 18, 2015
SOLUTIONS

Here are the warmup problems to try as everyone arrives.

1. There are two different right angle isosceles triangles that have a side with length 2. Draw both triangles and find their areas.

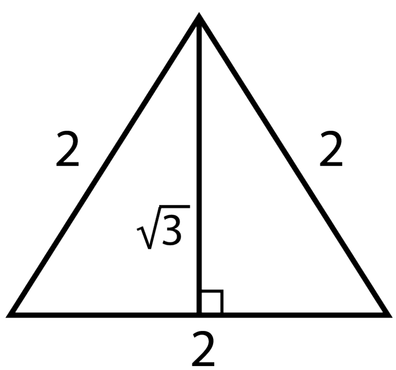




$$A = \frac{(2)(2)}{2} = 2$$

$$A = \frac{(\sqrt{2})(\sqrt{2})}{2} = 1$$

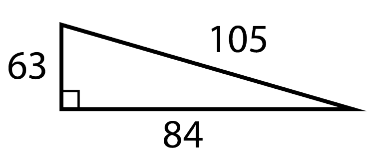
2. An equilateral triangle has side length 2. Find the area of this triangle.



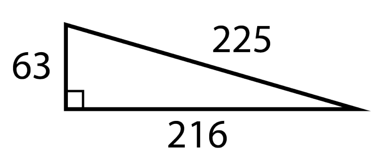
$$A = \frac{(2)(\sqrt{3})}{2} = \sqrt{3}$$

3. There is a right angled triangle with one side 63 and the other two sides both have integer lengths. Draw the triangle and find the lengths of all 3 sides.
   (Can you find other such triangles?)

Think 3, 4, 5:



Think 7, 24, 25:



Think about $2ab$, $a^2 - b^2$, and $a^2 + b^2$.

Let $a = 8$, $b = 1$ then
$2ab = 16$.
$a^2 - b^2 = 63$, and
$a^2 + b^2 = 65$.

Special Triangles

We should know the definitions and the properties of the following triangles.

a) Scalene

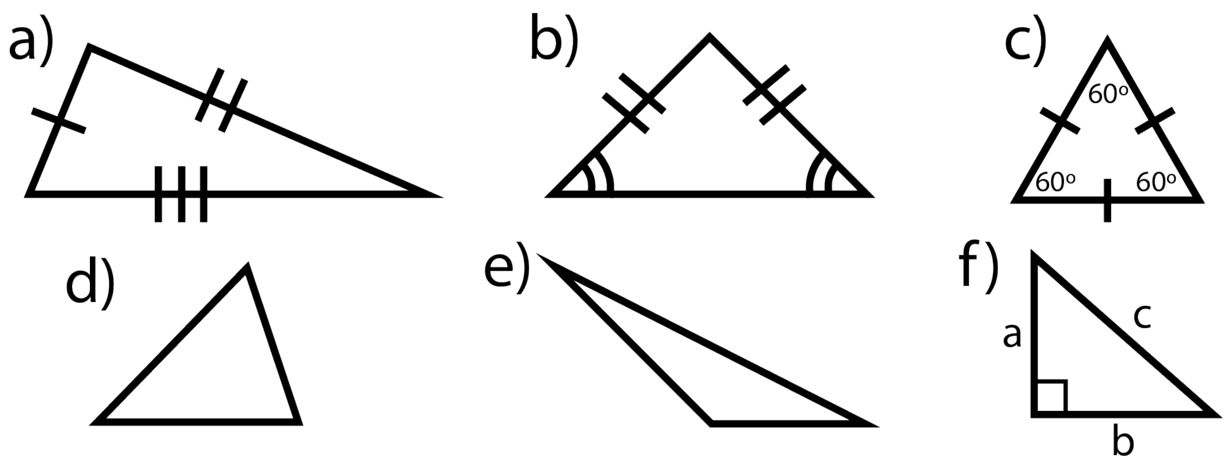
b) Isosceles

c) Equilateral

d) Acute

e) Obtuse

f) Right triangles

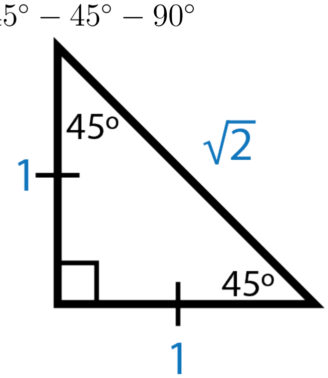


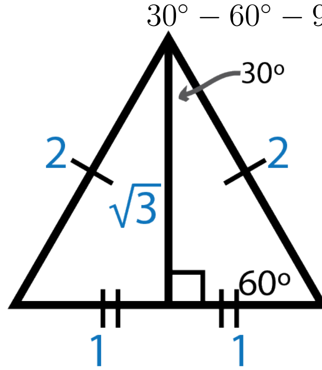
all angles less than 90° | one angle greater than 90° | $c^2 = a^2 + b^2$

The sum of the interior angles in any triangle is 180°.
Triangles have 3 sides.

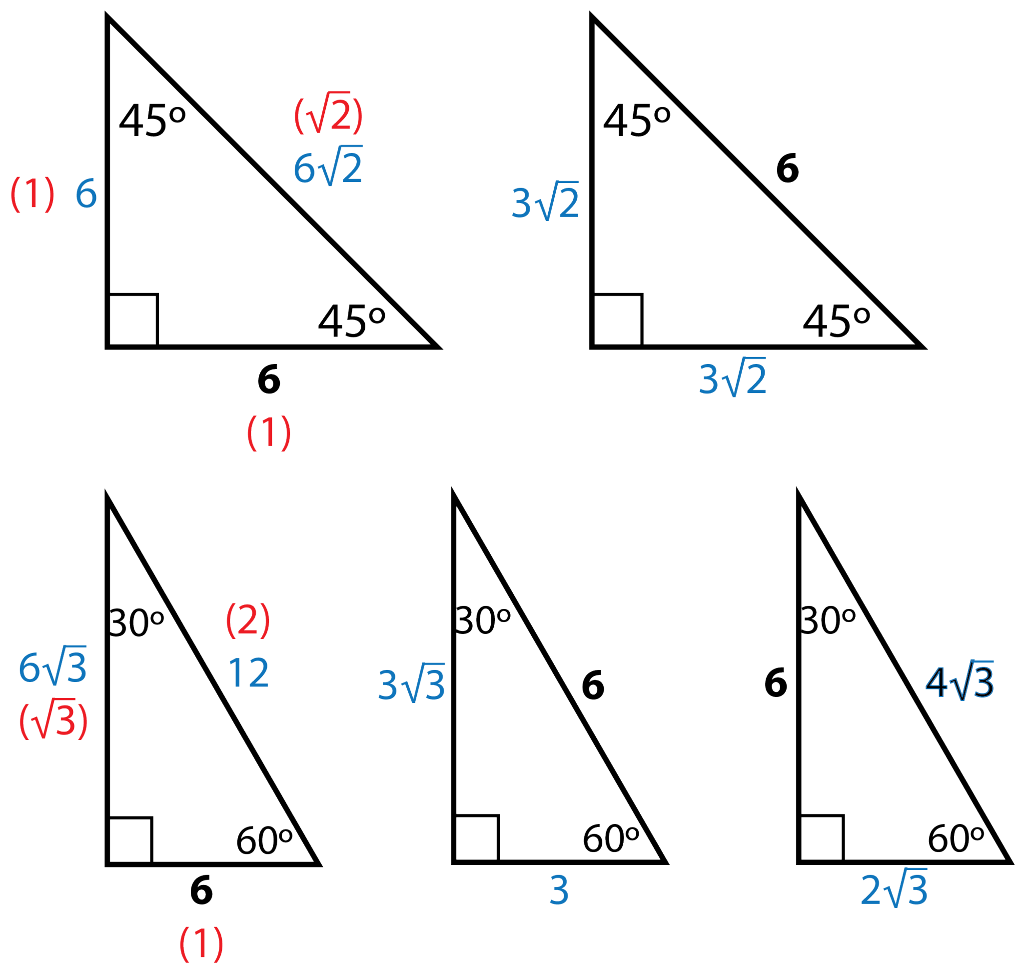
There are two more Special Triangles we should know.

$45° - 45° - 90°$ and $30° - 60° - 90°$

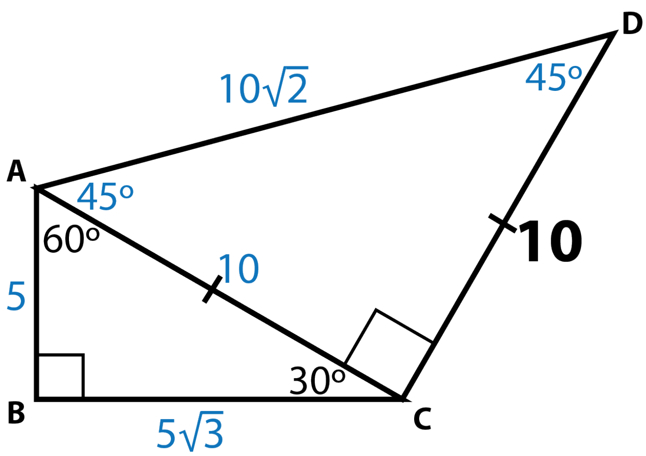




Example 1.
Find all the sides for any $45° - 45° - 90°$ or $30° - 60° - 90°$ triangles that have one side length 6.



Example 2.
Find the Perimeter and Area of ABCD.



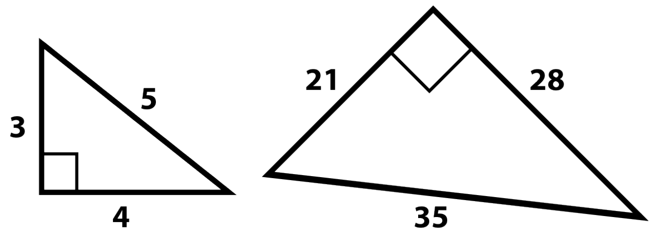
$$\begin{aligned} Perimeter &= 5 + 5\sqrt{3} + 10 + 10\sqrt{2} \\ &= 15 + 10\sqrt{2} + 5\sqrt{3} \end{aligned}$$

$$\begin{aligned} Area &= Area_{ABC} + Area_{ACD} \\ &= \frac{(5\sqrt{3})(5)}{2} + \frac{(10)(10)}{2} \\ &= \frac{25\sqrt{3}}{2} + 50 \end{aligned}$$

Pythagorean Triples

Right angled triangles that have whole number sides have sides that are called Pythagoerean Triples.

The most recognized is our 3,4,5 triangle.



You should also know the next two or three smallest triples. 5,12,13 7,24,25 and 8,15,17.

There is also a cool formula that allows you generate all sorts of Pythagorean Triples.

Take any two whole numbers a and b where $a > b$.

A Pythagorean Triple is formed by $2ab, a^2 - b^2, a^2 + b^2$.

For example.......

Let $a = 7$, and $b = 3$.

$$2ab = 2(7)(3) = 42 \qquad a^2 - b^2 = 7^2 - 3^2 = 40 \qquad a^2 + b^2 = 7^2 + 3^2 = 58$$

Can we show algebraically that $2ab, a^2 - b^2, a^2 + b^2$ always fits the Pythagorean Theorem?

We want to show that the left hand side (LHS) is equal to the right hand side (RHS). In other words, we want to show $LHS = (a^2 + b^2)^2 = (2ab)^2 + (a^2 - b^2)^2 = RHS$.

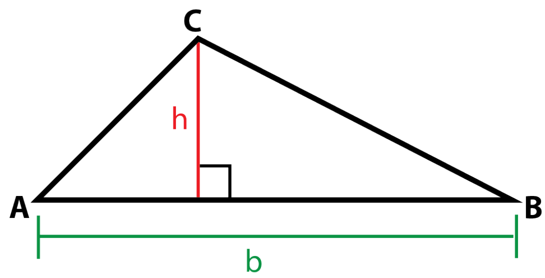
$$\begin{aligned} LHS &= (a^2 + b^2)^2 \\ &= (a^2 + b^2)(a^2 + b^2) \\ &= a^4 + 2a^2b^2 + b^4 \end{aligned}$$

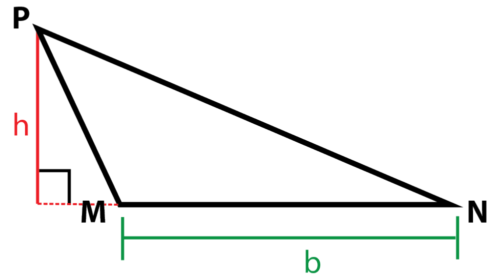
$$\begin{aligned} RHS &= (2ab)^2 + (a^2 - b^2)^2 \\ &= 4a^2b^2 + (a^2 - b^2)(a^2 - b^2) \\ &= 4a^2b^2 + a^4 - 2a^2b^2 + b^4 \\ &= a^4 + 2a^2b^2 + b^4 \end{aligned}$$

Observe that both sides give the same result, therefore we have that $LHS = RHS$, as required.

Triangle Area

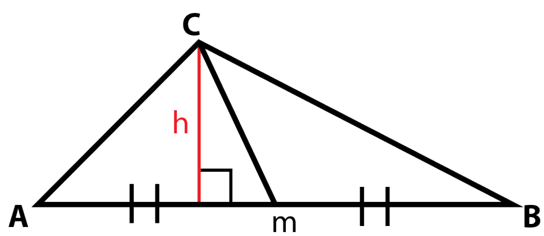
We certainly know the formula $A = \frac{bh}{2}$ or $A = \frac{1}{2}bh$.
Please remember that the height,"h", may be inside or outside the triangle.

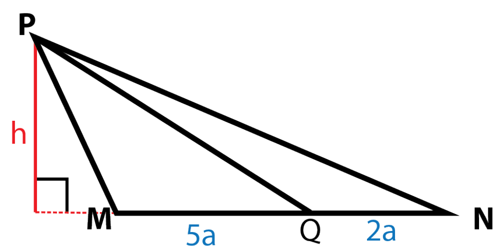




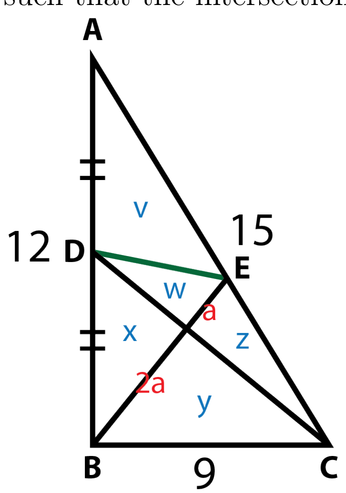
You could also spend some time to learn or look up Pic's Theorem to find Triangle area.

An important property that is often used in Contest Mathematics comes from splitting triangles up.





Example: In $\triangle ABC$, AB=12, BC=9 and AC=15. D is the midpoint of AB. E is a point on AC such that the intersection of DC and BE splits BE into the ratio 2:1. Find the area of $\triangle ADE$.



Let the areas be v, w, x, y, and z, as shown.

$$x + y = 27 \qquad (1)$$
$$v + w + x = 27 \qquad (2)$$
$$y = 2z \qquad (3)$$
$$x = 2w \qquad (4)$$

Substitute (3) and (4) in (1) $\quad (2w) + (2z) = 27$

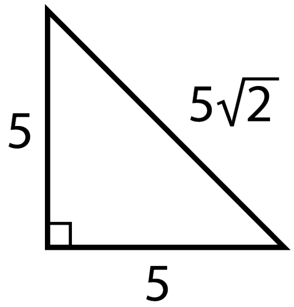
$$2(w + z) = 27$$

Substitute (2) $\quad 2(27 - v) = 27$

$$54 - 2v = 27$$
$$27 = 2v$$
$$13.5 = v = Area_{\triangle ADE}$$

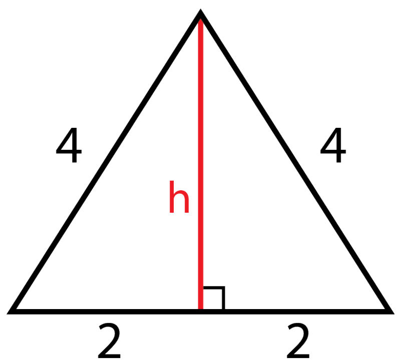
Problem Set

1. The perimeter of a right isosceles triangle is $10 + 5\sqrt{2}$ cm. What is the area of the triangle.



$$A = \frac{(5)(5)}{2} = \frac{25}{2}\ \text{cm}^2$$

2. An equilateral triangle has side length 4m. What is the area of the triangle.



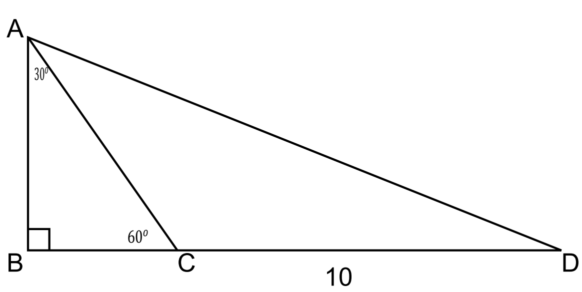
$$h = 2\sqrt{3}\ \text{m}$$

$$A = \frac{(4)(2\sqrt{3})}{2} = 4\sqrt{3}\ \text{m}^2$$

3. I believe that there are 12 different Pythagorean Triples which contain a 60. List as many of them as you can find.

From 3, 4, 5	From 5, 12, 13	From $2ab$, $a^2 - b^2$, $a^2 + b^2$
60, 80, 100	60, 144, 156	Let $a = 30$, $b = 1$ to get 60, 899, 901
45, 60, 75	25. 60, 65	Let $a = 8$, $b = 2$ to get 32, 60, 68
36, 48, 60		Try more of these or check the internet!

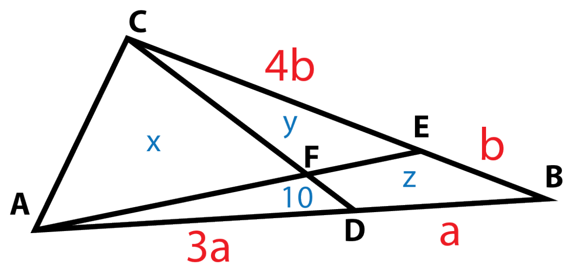
4. The area of $\triangle ACD$ is 30. CD=10. What is the area of $\triangle ABD$?



AB is the height of $\triangle ACD$. Then $AB = 6$ to get an area of 30. Use the special triangles to get $BC = 2\sqrt{3}$. Therefore,

$$A = \frac{(10 + 2\sqrt{3})(6)}{2} = 30 + 6\sqrt{3}$$

5. $\triangle ABC$ has area 300. D divides AB in the ratio 3:1. (i.e. AD:DB=3:1) E divides BC in the ratio 1:4. AE and CD intersect at F. The area of $\triangle AFD$ is 10. Find the area of quadrilateral BDFE.



Join DE and label the areas 10, x, y, and z as shown.

For $\triangle ABC$ split by CD:

$x + 10 = 3(y + z)$ — (1)

For $\triangle ABC$ split by AE:

$x + y = 4(z + 10)$ — (2)

For the whole area of $\triangle ABC$:

$$\begin{aligned} x + y + z + 10 &= 300 \\ x + y + z &= 290 \\ \text{Substitute (2)}: \quad [4(z + 10)] + z &= 290 \\ 4z + 40 + z &= 290 \\ 5z &= 250 \\ z &= 50 \end{aligned}$$

Therefore, the area of $BDEF$ is 50.