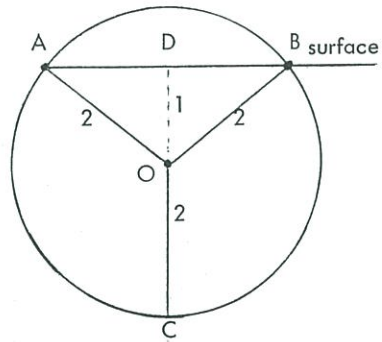
Official Solutions for Wed. Nov. 25, 2015

JMC 1977 #22

Solution.

The diagram represents a cross section of the tank.



The distance AB is twice

$BD = 2\sqrt{2^2 - 1^2} = 2\sqrt{3}$.

Surface area = area of a rectangle of dimensions 16 and $2\sqrt{3}$, that is, $32\sqrt{3}$.

The answer is (A).

JMC 1981 #24

Solution

In 90! there are 45 factors which are multiples of 2. Of these 22 are multiples of 4, 11 are multiples of 8, 5 are multiples of 16, 2 are multiples of 32, and 1 is a multiple of 64. Then the exponent of the highest power of 2 is $45 + 22 + 11 + 5 + 2 + 1 = 86$. (A)

JMC 1976 #18

Solution 1.

The line has equation

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Since (2, 1) lies on the line,

$$\frac{2}{a} + \frac{1}{b} = 1,$$

that is,

$$2b + a = ab,$$

$$2b = ab - a,$$

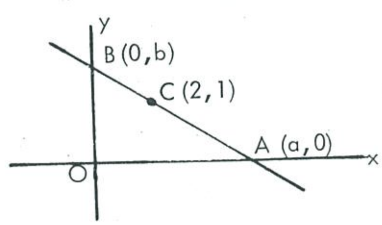
$$2b = a(b - 1).$$

The answer is A.

Solution 2.

$$\text{Slope } AC = \frac{1 - 0}{2 - a};$$

$$\text{slope } BC = \frac{1 - b}{2 - 0}.$$



Equating slopes, we obtain

$$\frac{1}{2 - a} = \frac{1 - b}{2},$$

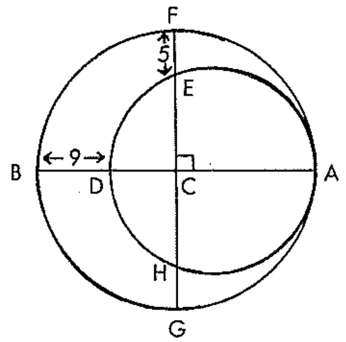
$$2 = 2 - a - 2b + ab,$$

$$2b = a(b - 1).$$

The answer is A.

JMC 1974 #29

Solution 1.



Let $DC = x$, and use the theorem on intersecting chords.

$BC = CA = CG = EF = 9 + x$

= radius of larger circle.

From the small circle,

$$DC \cdot CA = EC \cdot CH.$$

$$x(9+x) = (9+x-5)^2$$

$$9x + x^2 = (4+x)^2 = x^2 + 8x + 16$$

By transposition, $x = 16$.

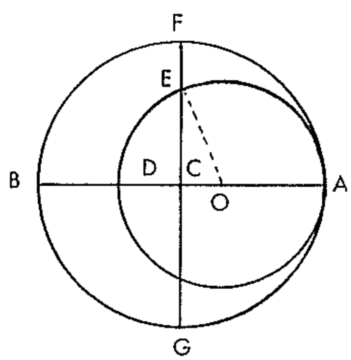
Hence the required diameter

$= DC + CA$

$= 16 + (9 + 16) = 41.$

The answer is (C).

Solution 2.



Let O be the centre of the smaller circle. Designate the radii of the larger and smaller circles by R and r respectively.

$\therefore \quad 2R = 2r + 9$

Hence $R = r + \frac{9}{2}$.

Now $CE = R - 5$

$= r + \frac{9}{2} - 5$

$= r - \frac{1}{2}$

and $OC = R - r$

$= r + \frac{9}{2} - r$

$= \frac{9}{2}$

In the right triangle ECO,

$$EO^2 = CE^2 + OC^2$$

$$r^2 = (r - \tfrac{1}{2})^2 + (\tfrac{9}{2})^2$$

$$= r^2 - r + \frac{1}{4} + \frac{81}{4}$$

$$\therefore r = \frac{41}{2}$$

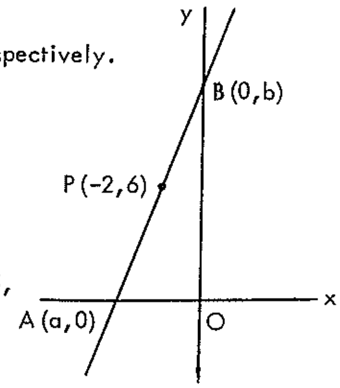
Hence the required diameter is $2r = 41$.

The answer is (C).

JMC 1981 #26

Solution

Let the x and y intercepts of the line be a and b respectively.



Then slope AP = slope AB

$$\frac{6}{-2-a} = \frac{b}{-a}$$

$\therefore \quad b = \dfrac{6a}{a+2}$

Since the area of ΔAOB is 25,

then $\frac{1}{2}ab = 25$.

$\therefore \; a\left(\dfrac{6a}{a+2}\right) = 50$

$6a^2 + 50a + 100$

$3a^2 - 25a - 5- = 0$

$(3a+5)(a-10) = 0$

$a = -\frac{5}{3}$ or 10.

The correct answer is (C).

JMC 1975 #26

Solution 1. $\dfrac{x-18}{x^2-x-6} = \dfrac{P}{x+2} + \dfrac{Q}{x-3}$

$$= \frac{P(x-3)+Q(x+2)}{x^2-x-6}$$

$$= \frac{(P+Q)x+(2Q-3P)}{x^2-x-6}$$

Since this is an identity,

$x - 18 \equiv (P+Q)x + (2Q-3P)$.

Hence $2Q - 3P = -18$,

$Q + P = 1$.

Solving, we get $P = 4$, $Q = -3$.

Hence $P - Q = 4 - (-3) = 7$.

Solution 2. As in Solution 1,

$x - 18 \equiv P(x-3) + Q(x+2)$

Since this identity holds for all x, it holds for $x = 3$.

Thus $-15 = 5Q$, $Q = -3$.

Also, the identity holds for $x = -2$;

hence $-20 = -5P$, $P = 4$.

As before, $P - Q = 7$.

PASCAL 1982 #25

Solution

List the integers as follows:

000 000
000 001
000 002
⋮
999 998
999 999

There are 6 000 000 digits in this list and the digits 0, 1, 2, ... , 9 each appear an equal number of times. So

JMC 1976 #26

Solution

There are 5 (4) (3) = 60 possible numbers.

By symmetry, each digit must appear $\frac{60}{5} = 12$ times

in each of the first, second, and third positions.

So the digits in each position add to 12 (2 + 3 + 4 + 5 + 6) = 240

Units digits give 240.

Tens digits give 2400.

Thousands digits give 24000.

Total sum is 26,640.

The answer is D.

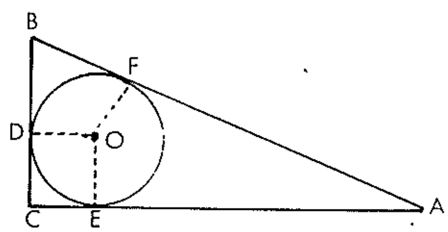
Note: The 60 numbers are

234	245	345	456
235	246	346	
236	256	356	

and all rearrangements of these (234 gives itself, 243, 342, 324, 432, and 423).

JMC 1976 #23

Solution 1.



Since $29^2 = 21^2 + 20^2$, we find that C is a right angle.

Let BF = BD = x,

DC = CE = r, FA = EA = y. (tangents from external points are equal)

Then r + x = 20, r + y = 21, x + y = 29.

Thus 2r + x + y = 41, 2r = 12, r = 6 (and 2r = 12).

The answer is A.

Solution 2.

$\Delta BCA = \Delta BOC + \Delta OCA + \Delta OAB$.

$\frac{1}{2}(20)(21) = \frac{1}{2}r(20) + \frac{1}{2}r(21) + \frac{1}{2}r(29)$.

420 = r (70), r = 6 (and 2r = 12).

The answer is A.

(Note that one could get ΔBCA by Heron's formula without even noting that it is right-angled.)

JMC 1981 #30

Solution

Let $1+k=3a$, $1+2k=5b$, and $1+8k=7c$ where a,b,c, are integers.

$\therefore\ a=\frac{k+1}{3}$; $\therefore\ k=2,5,8,11,\ldots,59,62,65,\ldots$.

$b=\frac{2k+1}{5}$ $\therefore\ k=2,7,12,17,\ldots,57,62,67,\ \ldots$.

$c=\frac{8k+1}{7}$ $\therefore\ k=6,13,20,27,\ldots,55,62,69,\ \ldots$.

The smallest value of k satisfying all three conditions is 62. (D)

JMC 1979 #27

Solution

$$\frac{1}{(2)(3)}=\frac{1}{2}-\frac{1}{3},$$

$$\frac{1}{(3)(4)}=\frac{1}{3}-\frac{1}{4},$$

$$\cdots\cdots\cdots$$

$$\frac{1}{(61)(62)}=\frac{1}{61}-\frac{1}{62}.$$

Add to obtain $\frac{1}{2}-\frac{1}{62}=\frac{31}{62}-\frac{1}{62}=\frac{30}{62}=\frac{15}{31}$.

Thus $a=15$, $b=31$, $a+b=46$.

The answer is (D).