Intermediate Math Circles
November 04, 2015
Counting II
Problem Set Solutions

1. If 50 different students try out for a team of 30 players, in how many different ways can the coach choose the team?

   Solution

   Since the order of selection to the team does not matter, we simply choose 30 from 50 in $\binom{50}{30}$ ways.

   Some of you may have tried to determine the answer in expanded form using your calculator. The way that this answer is left is quite acceptable.

2. How many groups can be formed from 8 men and 5 women if:

   a.) the group must have exactly 2 women and 2 men?

   Solution

   There must be 2 men chosen from a total of 8 men and 2 women chosen from a total of 5 women. Using the product rule we obtain:

$$\begin{aligned}\binom{8}{2} \times \binom{5}{2} &= 28 \times 10 \\ &= 280\end{aligned}$$

   There are 280 ways to make a group that has exactly 2 women and 2 men.

   b.) the group can be any size, but must have at least one member and an equal number of men and women?

   Solution

   There can be a group with 1 woman and 1 man, or 2 women and 2 men, or 3 women and 3 men, or 4 women and 4 men, or 5 women and 5 men. Using the product and sum rules, we obtain:

$$\begin{aligned}&\binom{5}{1} \times \binom{8}{1} + \binom{5}{2} \times \binom{8}{2} + \binom{5}{3} \times \binom{8}{3} + \binom{5}{4} \times \binom{8}{4} + \binom{5}{5} \times \binom{8}{5} \\ =\ & (5)(8) + (10)(28) + (10)(56) + (5)(70) + (1)(56) \\ =\ & 40 + 280 + 560 + 350 + 56 \\ =\ & 1\ 286\end{aligned}$$

   There are 1 286 ways to make a group that has an equal number of men and women.

3. How many groups containing seven different numbers can be formed by selecting the numbers from the set $\{1, 2, \ldots, 20\}$ such that

a.) 19 is the largest number in the group?

Solution

If 19 is the largest number in the group, there are 6 more numbers that need to be put into the group from the 18 numbers that remain (as 20 and 19 cannot be put into the group). So there are $\binom{18}{6}$ or 18 564 groups.

$\therefore$ there are 18 564 ways to make a group of seven from this number set so that 19 is the largest number in the group.

b.) 9 is the middle number in the group?

Solution

If 9 is the middle number in the group, this means that there are 3 numbers from the 8 numbers that are less than 9 and 3 numbers from the 11 numbers that are greater than 9. So using the product rule, there are $\binom{8}{3} \times \binom{11}{3} = (56)(165) = 9\ 240$ groups.

$\therefore$ there are 9 240 ways to make a group of seven where 9 is the middle number in the group.

c.) the difference between the largest and smallest number in the group is equal to 14?

Solution

The possible pairs of largest and smallest numbers that differ by 14 are:

Smallest	**Largest**
1	15
2	16
3	17
4	18
5	19
6	20

There are 6 pairs of numbers from the set that differ by 14. For each of these pairs, there are 13 numbers in between. We must choose 5 numbers from the 13 to be put in the group. So there are $6 \times \binom{13}{5} = 6 \times 1\ 287 = 7\ 722$ groups.

There are 7 722 ways to make a group of seven in which the difference between the largest value and smallest value is 14.

4. With a standard deck of 52 cards, a subset of 5 cards is called a hand.

a.) How many hands are there?

Solution

Since the order of selection of the cards does not matter, we simply choose 5 cards from 52 cards in $\binom{52}{5}$ or 2 598 960 ways.

$\therefore$ there are 2 598 960 hands.

b.) How many hands contain exactly one pair? (2 of a kind and 3 different cards)

Solution

There are 13 different card values to choose for the one pair. Once the value is selected, there are four cards with the same numeric value and we can select two in $\binom{4}{2}$ ways. Once this pair is chosen, there are 12 remaining card values from which 3 different values must be chosen (giving us $\binom{12}{3}$ ways to choose our remaining 3 values). There are 4 cards (4 suits) with each value, giving us 4 ways to choose each of our 3 remaining cards. Since these four criteria must be followed at the same time, we can use the product rule.

$$\begin{aligned} 13 \times \binom{4}{2} \times \binom{12}{3} \times 4^3 &= 13 \times 6 \times 220 \times 64 \\ &= 1\ 098\ 240 \end{aligned}$$

$\therefore$ there are 1 098 240 hands that contain exactly one pair and three other cards, each with a different value.

c.) How many hands have 4 of a kind?

Solution

There are 13 different card values and we must choose one of them for our four of a kind. There is only one way to choose the four cards once the value is chosen. After choosing the four cards for our four of a kind, there are 48 cards remaining from which 1 card must be chosen for the final card.

Using the product rule, we obtain $\binom{13}{1} \times \binom{4}{4} \times \binom{48}{1} = 13 \times 1 \times 48 = 624$ different hands.

$\therefore$ there are 624 hands that contain 4 of a kind.

5. How many **permutations** of the numbers $1, 2, \ldots,\ 20$ taken 7 at a time

a.) contain 3 odd and 4 even numbers?

Solution

There are 3 odd numbers that must be chosen from a total of 10 odd numbers and 4 even numbers that must be chosen from a total of 10 even numbers. Since order matters, we must arrange the numbers once chosen. This can be done in 7! ways. Using the product rule there are:

$$\begin{aligned}\binom{10}{4} \times \binom{10}{3} \times 7! &= 210 \times 120 \times 5\ 040 \\ &= 127\ 008\ 000 \text{ permutations.}\end{aligned}$$

$\therefore$ there are 127 008 000 permutations from this set that contain 3 odd and 4 even numbers.

b.) contain 2 single-digit numbers and 5 two-digit numbers?

Solution

There are 2 numbers that must be chosen from a total of 9 single-digit numbers and 5 numbers that must be chosen from a total of 11 two-digit numbers. Since order matters, we must arrange the numbers once chosen. This can be done in 7! ways. Using the product rule there are:

$$\begin{aligned}\binom{9}{2} \times \binom{11}{5} \times 7! &= 36 \times 462 \times 5\ 040 \\ &= 83\ 825\ 280 \text{ permutations.}\end{aligned}$$

$\therefore$ there are 83 825 280 permutations with this set that contains 2 single-digit numbers and 5 two-digit numbers.

c.) have the 7 numbers arranged in increasing order?

Solution

For this question, the 7! does not need to be taken into account as each number can only go in one place since all the numbers must be in increasing order. So we choose seven numbers from twenty numbers in $\binom{20}{7} = 77\ 520$ ways.

$\therefore$ there are 77 520 ways to arrange the numbers in increasing order.

6. Evaluate the following **without** the use of a calculator.

a) $\dfrac{\binom{7}{3}}{\binom{7}{4}}$ b) $\dfrac{\binom{12}{8}}{\binom{9}{4}}$ c) $\dfrac{\binom{n}{3}}{\binom{n}{2}}$, $n \geq 3$.

Solution

a)

$$\frac{\binom{7}{3}}{\binom{7}{4}} = \frac{\left(\dfrac{7!}{3! \times 4!}\right)}{\left(\dfrac{7!}{4! \times 3!}\right)} = \frac{7!}{3! \times 4!} \times \frac{4! \times 3!}{7!} = 1$$

b)

$$\frac{\binom{12}{8}}{\binom{9}{4}} = \frac{\dfrac{12!}{8! \times 4!}}{\dfrac{9!}{4! \times 5!}} = \frac{12!}{8! \times 4!} \times \frac{5! \times 4!}{9!} = \frac{12 \times 11 \times 10}{8 \times 7 \times 6} = \frac{11 \times 5}{2 \times 7} = \frac{55}{14}$$

Note: $\dfrac{12!}{9!} = \dfrac{12 \times 11 \times 10 \times 9!}{9!} = 12 \times 11 \times 10$ and $\dfrac{5!}{8!} = \dfrac{5!}{8 \times 7 \times 6 \times 5!} = \dfrac{1}{8 \times 7 \times 6}$

c)

$$\frac{\binom{n}{3}}{\binom{n}{2}} = \frac{\dfrac{n!}{3! \times (n-3)!}}{\dfrac{n!}{2! \times (n-2)!}} = \frac{n!}{3! \times (n-3)!} \times \frac{2! \times (n-2)!}{n!} = \frac{n-2}{3}$$

Note: $\dfrac{(n-2)!}{(n-3)!} = \dfrac{(n-2) \times (n-3)!}{(n-3)!} = (n-2)$ and $\dfrac{2!}{3!} = \dfrac{2!}{3 \times 2!} = \dfrac{1}{3}$