Intermediate Math Circles
October 14, 2015
Rates

A rate is a comparison of two quantities that have different units of measure. Many times one of the quantities involves time. The other quantity could be a distance, a volume, a number of items or many other things.

For example 100 $\frac{\text{km}}{\text{h}}$, 15 L/minute, and 15 cars per day are all rates.

Rate problems involving time can be solved with one basic equation:

$$RATE = \frac{QUANTITY}{TIME}$$

This equation can be rearranged to solve for any value given the other two values.

Example:

A hose expels water at a flow rate of 15 L/min for 5 minutes. How much water does it put out in this time?

Sol$^{\underline{\text{n}}}$: Let w represent the amount of water.

Then $15\frac{\text{L}}{\text{min}} = \frac{w}{5\ \text{min}}$, so $w = (15\ \frac{\text{L}}{\text{min}})(5\ \text{min}) = 75$ L.

Therefore 75 L of water flows out of the hose in 5 minutes..

Often times in rate problems there are two objects moving at different rates and we may want to figure out the rate at which they are moving apart.

Example:

Norma leaves her house travelling east at 80 $\frac{\text{km}}{\text{h}}$ at the same time that Kelly leaves the house travelling west at 60 $\frac{\text{km}}{\text{h}}$. How far apart are they after one hour? After two hours?

Sol$^{\underline{\text{n}}}$:

Let n be the distance Norma travels in an hour and k be the distance Kelly travels in an hour.

$$\begin{aligned} n &= \left(80\ \tfrac{\text{km}}{\text{h}}\right)(1\ \text{h}) \\ &= 80\ \text{km} \end{aligned} \qquad \begin{aligned} k &= \left(60\ \tfrac{\text{km}}{\text{h}}\right)(1\ \text{h}) \\ &= 60\ \text{km} \end{aligned}$$

Therefore, after one hour they are 80 km + 60 km = 140 km apart.

After two hours they will be $2 \times 140 = 280$ km apart.

The previous example illustrates the basic concept that when Object A is moving at rate a from a certain point and Object B is moving at rate b in opposite directions from the same point, their rate of separation is $a + b$.

If Object A and Object B move in the same direction, their rate of separation would be $|a - b|$. The bars denote the absolute value of $(a - b)$, the positive value. If $a - b = -5$ then $|a - b| = 5$. If $a - b = 5$ then $|a - b| = 5$.

Example:

A horse gallops away from a barn at 15 $\frac{\text{m}}{\text{s}}$ at which time a human immediately starts running after it at 8 $\frac{\text{m}}{\text{s}}$. How far ahead will the horse be after 5 seconds? After how long will the horse be 100 m ahead?

Sol$^{\underline{\text{n}}}$: The rate of separation of the horse and the human is $\left|8\ \frac{m}{s} - 15\ \frac{m}{s}\right| = \left|-7\ \frac{m}{s}\right| = 7\ \frac{m}{s}$

Therefore, after 5 seconds the horse will be$(7\ \frac{\text{m}}{\text{s}})(5\text{ s}) = 35$ m ahead.

Let t be the time at which the horse will be 100 m ahead.

$$7\ \frac{\text{m}}{\text{s}} = \frac{100\text{ m}}{t}$$

If we rearrange the equation:

$$t = \frac{100\text{ m}}{7\ \frac{\text{m}}{\text{s}}}$$

$$\doteq 14.3\text{ s}$$

Therefore, after approximately 14.3 seconds the horse will be 100 metres ahead.

Another type of problem involves the rate at which several people can complete an activity when working together compared to when the activity is completed alone.

Example:

A long driveway needs to be shovelled. Al knows he can finish shovelling it in 3 hours, Bob can finish it in 4 hours, Carl can finish it in 6 hours and Doug can finish it in 8 hours. If they all work together, how long will it take them to finish shovelling the driveway?

Sol$^{\underline{\text{n}}}$:
In one hour Al shovels $\frac{1}{3}$ of the driveway, Bob shovels $\frac{1}{4}$, Carl shovels $\frac{1}{6}$ and Doug shovels $\frac{1}{8}$.

All together, in one hour, they shovel $\frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} = \frac{8}{24} + \frac{6}{24} + \frac{4}{24} + \frac{3}{24} = \frac{21}{24} = \frac{7}{8}$ of the driveway.

Let t be the amount of time it takes them to shovel the entire driveway.

$$\frac{\frac{7}{8}}{1\text{ h}} = \frac{1}{t}$$

$$\frac{7}{8}t = 1$$

$$t = \frac{8}{7} \doteq 68.6$$

Therefore together they can shovel the driveway in approximately 69 minutes.

Example:

Jack and John are supposed to blow up balloons for a party. Jack knows that it will take him 8 hours to blow up the balloons by himself, and John knows that it will take him 6 hours to blow up the balloons by himself. They plan to work together to get it done faster, but John shows up late to help and Jack has already blown up half of the balloons. What was the total amount of time it took to blow up all the balloons.

Sol$^{\underline{n}}$:

Since Jack takes 8 hours to blow up all the balloons, it must have taken him 4 hours to blow up half of the balloons.

Jack can blow up $\frac{1}{8}$ of the balloons in an hour, and John can blow up $\frac{1}{6}$ of the balloons in an hour. Together, they can blow up $\frac{1}{8}+\frac{1}{6}=\frac{3}{24}+\frac{4}{24}=\frac{7}{24}$ of the balloons in one hour.

Let t be the time, in hours, it takes them to blow up half of the balloons.

$$\frac{\frac{7}{24}}{1\text{ h}}=\frac{\frac{1}{2}}{t}$$

$$\frac{7}{24}t=\frac{1}{2}$$

$$t=\frac{12}{7}$$

Therefore, together it took $4+\frac{12}{7}=5\frac{5}{7}$ hours, approximately 5 hours and 43 minutes.

From the Problem Set

5. Two candles of equal length are lit at noon. One candle takes 9 hours to completely burn while the other takes 6 hours to completely burn. At what time will the slower burning candle be exactly twice as long as the faster burning one?

   Let L represent the original length of each candle.

   Let t represent the time, in hours, until the slower burning candle is twice the height of the faster burning candle.

   The faster candle burns completely in 6 hours so in 1 hour, $\frac{1}{6}$ of the candle burns. In t hours, $\frac{1}{6}tL$ burns leaving $L - \frac{1}{6}tL$ of the candle. Similarly, the slower candle burns completely in 9 hours so in 1 hour, $\frac{1}{9}$ of the candle burns. In t hours, $\frac{1}{9}tL$ burns leaving $L - \frac{1}{9}tL$ of the candle.

   We want to know the time when the length of the slower burning candle is twice the length of the faster burning candle.

$$\begin{array}{rrcl} & L - \dfrac{1}{9}tL & = & 2 \times (L - \dfrac{1}{6}tL) \\ & L - \dfrac{1}{9}tL & = & 2L - \dfrac{1}{3}tL \\ \text{Dividing by } L \text{ since } L > 0, & 1 - \dfrac{1}{9}t & = & 2 - \dfrac{1}{3}t \\ \text{Multiplying by 9,} & 9 - t & = & 18 - 3t \\ & 2t & = & 9 \\ & t & = & 4.5 \text{ h} \end{array}$$

   $\therefore$ the slower burning candle will be double the height of the faster burning candle in 4.5 hours. Since they started burning at noon, this will occur at 4:30 p.m.