Intermediate Math Circles
October 21, 2015
Rates II

Last week, we looked at problems that involved finding combined rates when two or three people worked on an action together. We can generalize this.

If there are n people and person i (with $1 \leq i \leq n$) takes a_i hours to complete the action on their own, then in one hour that person can complete $\frac{1}{a_i}$ of the action. Let T be the amount of time it takes all n people together to complete the action.

The following general result allows us to calculate the total amount of time required.

$$\frac{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}{1\,\text{hour}} = \frac{1}{T}$$

Using this formula we can solve for the amount of time it takes any individual to complete the action or the amount of time it takes all of the people together.

Acceleration:

Acceleration is the rate at which a moving object slows down or speeds up. This quantity measures a change in speed against time.

$$Acceleration = \frac{Change\ in\ Speed}{Time}$$

A change in speed has the same units as speed. Some possibilities for units include $\frac{\text{m}}{\text{s}}$ or $\frac{\text{km}}{\text{h}}$.

As a result, the units of acceleration may be $\frac{\text{m}}{\text{s}^2}$ or $\frac{\text{km}}{(\text{h})(\text{s})}$. This may seem odd, but you can think of it as $\frac{\text{m}}{\text{s}}$ per second or $\frac{\text{km}}{\text{h}}$ per second.

A change in speed is equal to the final speed of the object minus the initial speed of the object.

If an object is speeding up, its final speed will be greater than its initial speed, so its change in speed and its acceleration will be positive.

If an object is slowing down, its final speed will be less than its initial speed, so its change in speed and its acceleration will be negative.

Example

If a car starts at 10 $\frac{\text{km}}{\text{h}}$ and accelerates at $5\frac{\text{km}}{(\text{h})(\text{s})}$ for 10 seconds. What is its final speed?

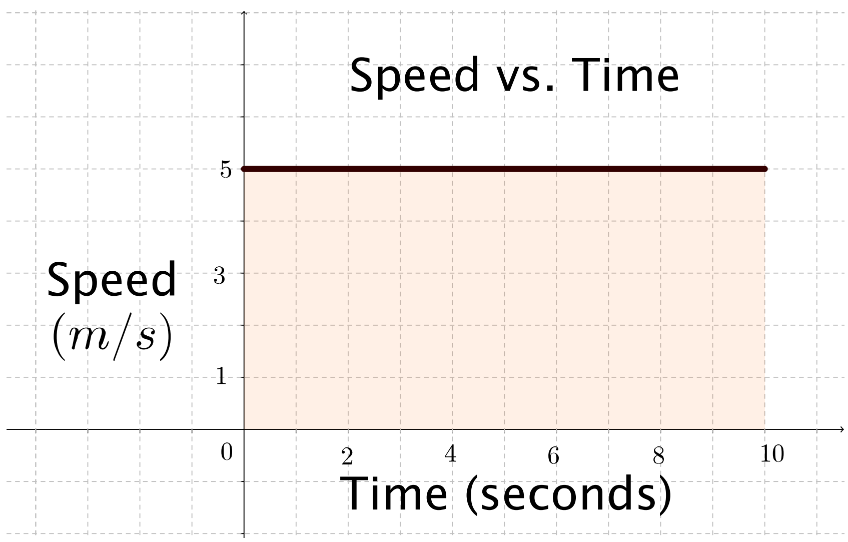
Solution:
Let $\triangle v$ represent the change in speed. Using $Acceleration = \dfrac{Change\ in\ Speed}{Time}$

$$5\ \frac{\text{km}}{(\text{h})(\text{s})} = \frac{\triangle v}{10\ \text{s}}$$

$$\triangle v = 5\ \frac{\text{km}}{(\text{h})(\text{s})} \times 10\ \text{s} = 50\frac{\text{km}}{\text{h}}$$

Since the change in speed is 50 $\frac{\text{km}}{\text{h}}$ and the car's initial speed is 10 $\frac{\text{km}}{\text{h}}$,
its final speed is $10\frac{\text{km}}{\text{h}} + 50\frac{\text{km}}{\text{h}} = 60\frac{\text{km}}{\text{h}}$

Using acceleration and speed or time, you can find distance. Let us look at the case where there is 0 acceleration. The following is a speed-time graph of the situation:



You can see that there is a constant speed of 5 $\frac{\text{m}}{\text{s}}$ and the time is 10 s. Therefore, the distance is 5 $\frac{\text{m}}{\text{s}}$ $\times$ 10 s = 50 m.

Look at the shaded area under the line.

$$\begin{aligned} \text{Area} &= \text{Length} \times \text{Width} \\ &= 10\ \text{s} \times 5\ \frac{\text{m}}{\text{s}} \\ &= 50\ \text{m} \end{aligned}$$

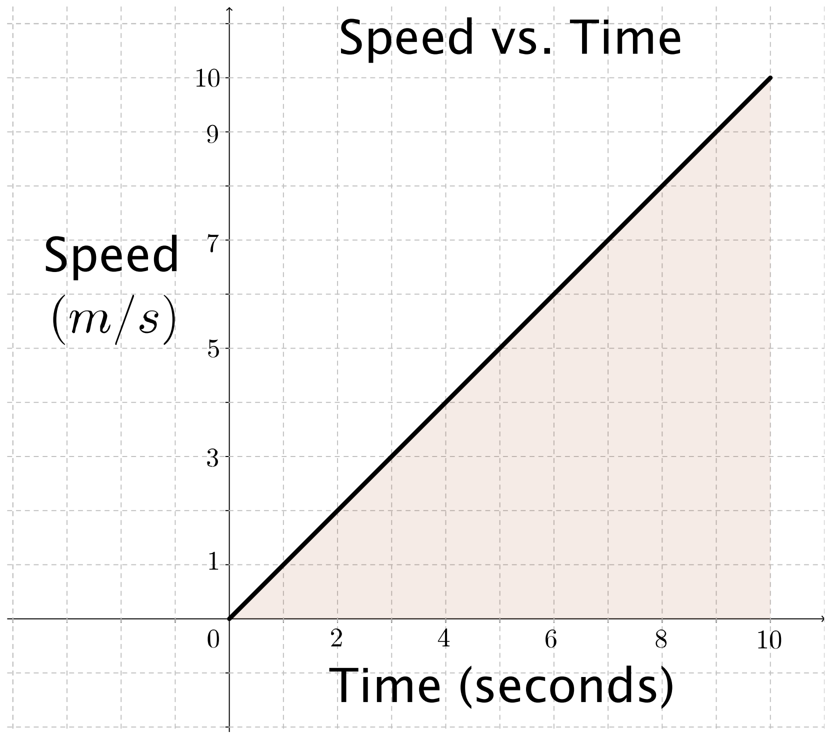
The numerical value associated with the area of the rectangle is the same as the numerical value associated with the distance travelled.

Let's now consider a situation where there is acceleration.

Example

An object starts at 0 $\frac{\text{m}}{\text{s}}$ and accelerates at 1 $\frac{\text{m}}{\text{s}^2}$ for 10 seconds. What distance does it travel?

Solution:



In this situation the speed increases at 1 $\frac{\text{m}}{\text{s}^2}$ for 10 seconds until it reaches 10 $\frac{\text{m}}{\text{s}}$ at 10 seconds.

The slope of this line represents the acceleration.

Determine the shaded area under the line using the formula for the area of a triangle.

$$\begin{aligned} \text{Area} &= \frac{1}{2}\text{ Base} \times \text{Height} \\ &= \frac{1}{2}(10\text{ s}) \times 10\ \frac{\text{m}}{\text{s}} \\ &= 50\text{ m} \end{aligned}$$

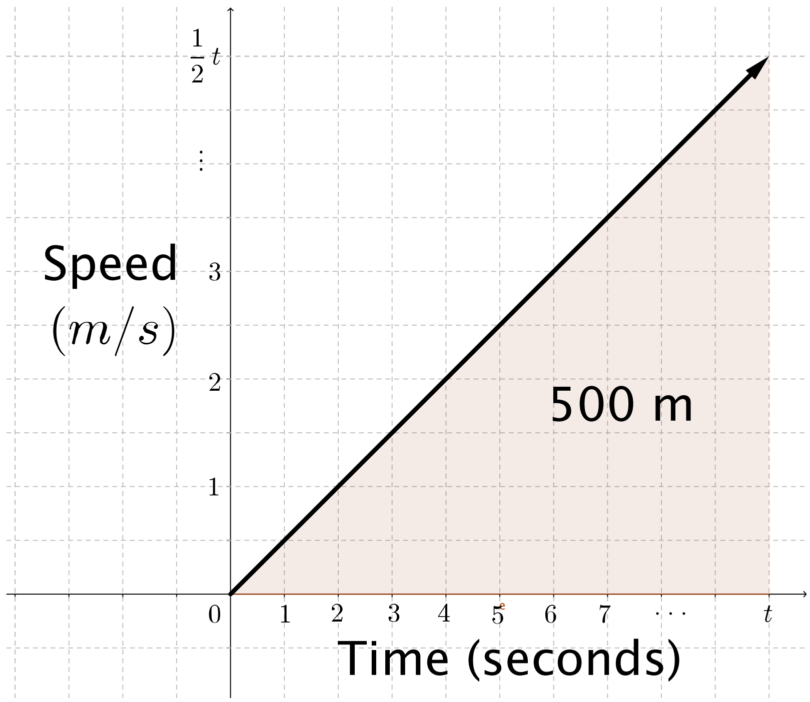
Again, the numerical value associated with the distance is equal to the numerical value associated with the area under the graph so the distance travelled is 50 m.

Regardless of the shape of the speed-time graph, the numerical value of the area underneath it is always equal to the numerical value associated with the distance travelled and the slope of the line at any time represents the acceleration at that time.

Example

A horse standing, at the start line of a race, begins to run around a 500 m track, accelerating at a rate of 0.5 $\frac{\text{m}}{\text{s}}$. How long does it take the horse to complete the lap?

Solution:



Let t be the time, in seconds, required to complete one 500 m lap of the track.

Then at t seconds, the speed will be $\frac{1}{2}t\ \frac{\text{m}}{\text{s}}$.

The distance is the shaded area under the line.

$$\begin{aligned} d &= \frac{1}{2} \times \text{Time} \times \text{Speed} \\ 500\text{ m} &= \frac{1}{2} \times (t\text{ s})\left(\frac{1}{2}t\ \frac{\text{m}}{\text{s}}\right) \\ 500 &= \frac{t^2}{4} \\ 2000 &= t^2 \\ \therefore\ t &= 20\sqrt{5}\text{ s}, \qquad t > 0 \end{aligned}$$

It takes the horse $20\sqrt{5}$ seconds, approximately 45 seconds, to complete one lap.