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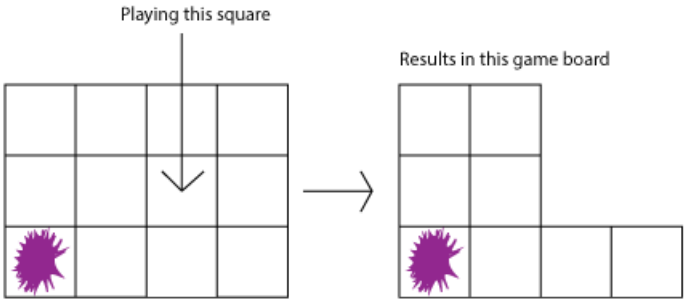
Centre for Education in
Mathematics and Computing

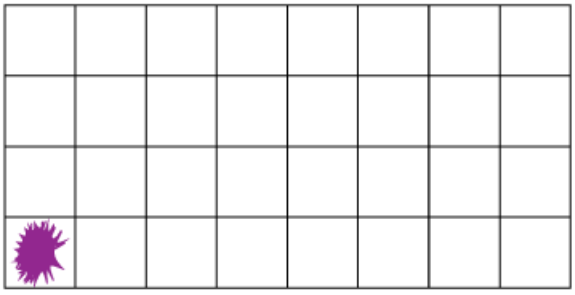
Grade 6 Math Circles
Combinatorial Games
November 3/4, 2015

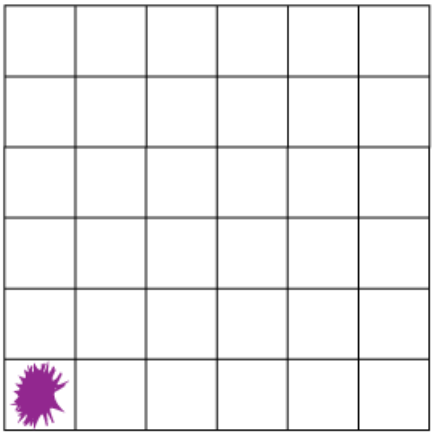
Chomp

Chomp is a simple 2-player game. There is a grid, and in the bottom-left corner of the grid is a poison square. Players alternate turns, and on each turn, the player selects a square and 'eats' that square and all of the squares above and to the right of it (see example below). The player who is forced to eat the poison square loses.



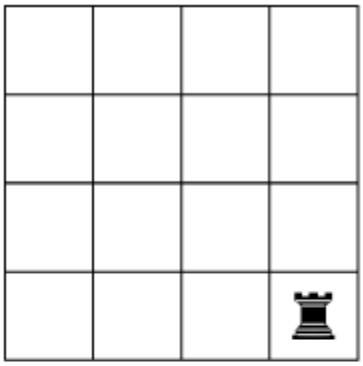
Try playing a few games of Chomp with the people around you using the game boards below. Change who goes first in each game. Is there one player who always wins? If so, why do you think that is?

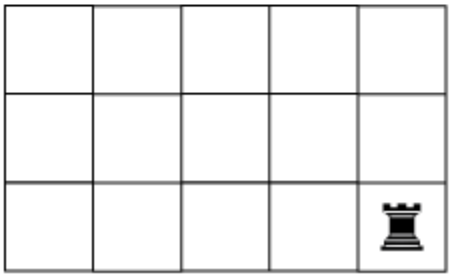


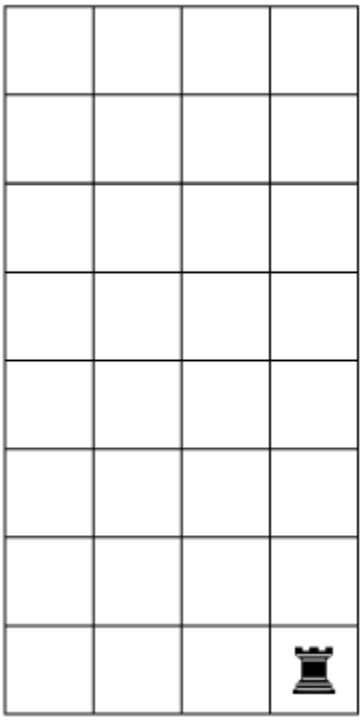


Rooks

Another simple game is called Rooks. In this game, a rook is placed in the bottom right corner of a rectangular board. Players take turns moving the rook either straight up any number of squares, or straight left any number of squares (but not both in one turn). The rook cannot move down or to the right. The player who places the rook in the top left corner of the board wins. Play a few games against those around you on the boards below.







Is there anything you noticed about this game? Are the certain points in the game where you know you will win?

Both Rooks and Chomp are examples of **impartial combinatorial games**. These are games are usually two-player games. For our purposes, these are games where any move available to one player is also available to another, there is perfect information (everyone knows everything about the game), there is a limited number of possible moves (so the game eventually ends), and nothing in the game is left to chance. usually, the first player who cannot move loses (or equivalently, the player who makes the last move wins).

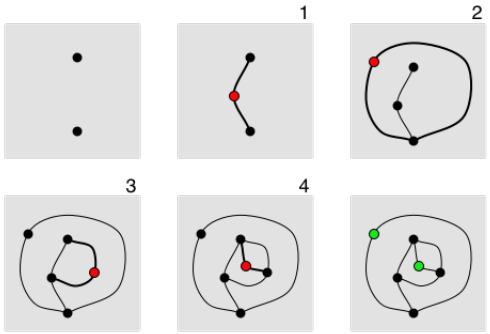
It has been shown that all games of this type can be simplified to a version of a special game called **Nim**. We will be exploring the game of Nim and developing a strategy for it.

Sprouts

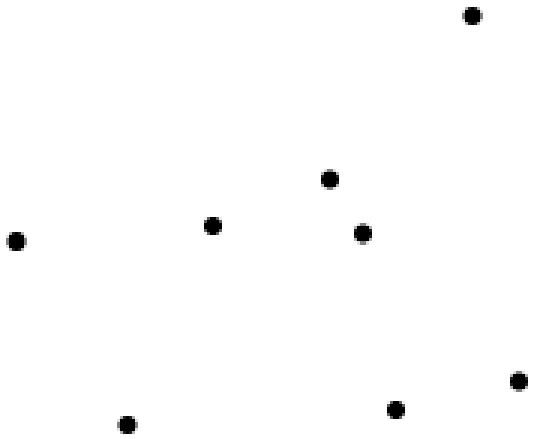
Sprouts is another 2-player game. In this game, the starting position consists simply of a set of dots. On each turn, a player draws a line between two dots, and adding a new dot somewhere on the line they just drew. There are only three rules for this game:

- The new line may be straight, or curved, but cannot touch or cross itself or any other line
- The new dot cannot be put on top of an existing dot. It must split the line into two shorter lines
- No dot may have more than three lines attached to it. A line from a dot to itself counts as two lines attached to the dot, and new dots have two lines already attached to them.

The first person who cannot make a new line according to the above rules loses. Below is an example of a game that starts with 2 dots.



In the space below, play a game of Sprouts with a friend.



Nim

Nim is a relatively simple game to play. There are a number of piles (or *heaps*) of chips. On each turn, a player removes chips from one of these piles. A player can only remove chips from one pile on each turn, and they must remove at least one chip. The player who takes the last chip(s) wins.

Nim Notation

We will denote each pile of chips in a Nim game as $*x$ where x is the number of chips in a pile. Games with multiple piles will be denoted as $*x_1 + *x_2 + \cdots + *x_n$ where each of the $*x_i$ represent a different pile.

For example, a game with one pile of 2 chips, and another pile of 3 chips would be written as $*2 + *3$. We write $*0$ to represent a game or pile with no chips.

We will call player 1 the player whose turn it is to move (i.e. the player who will make the next move)

Winning and Losing Games

In Nim, one player will always win, and one player will always lose. We know this because there are only so many chips on the board, and on each turn some chips are removed. This means that eventually all of the chips will be gone, and clearly only one person can pick up the last chip.

We says that a game is a **winning game** for a player if they can guarantee that they will win the game. This means that if the player makes a specific move on each of their turns, they will always win.

In contrast, a **losing game** is a game or position where a player will lose no matter what. In other words, a player can make any move, and the other player will still be able to win the game.

Exercise 1: What is the relationship between winning and losing games? If it possible for a game to be a winning game *and* a losing game for one player at the same time?

Exercise 2: Is $*0$ a winning or losing game for player 1? What about for player 2?

Basic Nim Strategies

Single Pile Nim

The simplest Nim games involve just a single pile of chips. This leads to a very simple strategy to win the game. Clearly, if there are no chips in the pile, then the player whose turn it is (we will call them player 1) loses. If there are chips in the pile, then player 1 can always win. All that they have to do is take all the chips. Then player 2 won't have any chips to take.

2-Heap Nim
The next simplest form of Nim is with two heaps. To examine games with two heaps, we will break our analysis into two cases: when the piles are the same size, and when they are different sizes.

2-Heaps of the same size - In this case both of the piles have the same number of chips. On each turn, a player can only remove chips from one pile. This leads to a simple *copycat* strategy that enables player 2 to always win. All player has to do is watch player 1 and then take the same number of chips as player 1 did, but take them from the other pile. That way, both piles will still have the same amount in them after each of player 2's turns.

2-Heaps of different sizes - In this scenario player 1 can exploit what we already know about Nim with two heaps of the same size. In that case, the player who moves second wins. Knowing this, player 1 can create a situation where there are two piles with the same amount of chips in it *and* where they move second in that instance. To do this they simply remove enough chips from the first pile to make the piles even. Then, the game is exactly like our previous case and the player can employ the copycat strategy.

Strategy in more complex Nim games
These strategies work in simple games. Can we determine who should win in more complex games?

For this Math Circle we will just show a couple of ways that we can see if player 1 should win or lose. This is not an exhaustive list, and just using these tools you may not be able to determine which player should win.

The tool we will use for this is again the *Copycat Principle*. Since a game with 2 piles of the same size is a losing game for player 1, we can actually just cross out any pairs of piles with the same number of chips.

For example, we could simplify the game $*1 + *5 + *1$ to $*5$ by crossing off the pair of ones. This simpler game with 1 pile is easy to know who wins - player 1, since they just need to take all of the chips from that pile. This means player 1 will win the original game, and their best move is to take all the chips from the pile of 5.

This method of simplifying also works in games with more than 3 piles. The game $*1 + *1 + *7 + *9 + *9 + *11$ could be simplified to $*7 + *11$ if we cross out pairs of numbers. This is also a winning game for player 1.

If, when we simplify the game we cross out every pile, then the game is a losing game for player 1. This is because we could split the original game into a number of smaller games with 2 piles of the same size. Player 2 then just needs to use the copycat principle in each of those smaller pairs.

For example, the game $*1 + *1 + *7 + *7 + *9 + *9$ is a losing game for player 1.

Converting other games to Nim values

It has been shown that all impartial combinatorial games are equivalent to a Nim game. We will look at converting the Rooks game to its Nim values.

The easiest square to convert to a Nim value is the ending square. If the rook is in the top left corner and it is your turn to move (i.e. your opponent just put it there) then you lost. Losing is equivalent to $*0$.

For each other square, we look at where it could move to. The Nim value is then the smallest number (at least 0) that is not in any of the squares that you could move to. For example, if we pick the square directly below the top left, then the only move is into the top left corner. This square has value $*0$ and so the smallest positive integer that we can't move to is 1. This means the Nim value for this square is $*1$. We can continue doing this for every square until we have the whole grid filled in. This results in the following Nim values for each square:

*0	*1	*2	*3
*1	*0	*3	*2
*2	*3	*0	*1
*3	*2	*1	*0

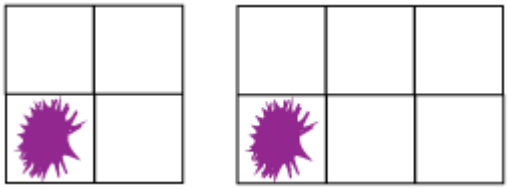
Problem Set

1. We said that an *impartial* game is a game where each player can make the exact same moves. Are chess and checkers impartial games? Why or why not?

2. Think back to the game Chomp from the beginning of this lesson. Who can always guarantee a win if it is played on a 2x2 board? What about a 3x2 board? (Boards are illustrated below).

   (a) Play a few games with a friend on each board to determine who you think should win

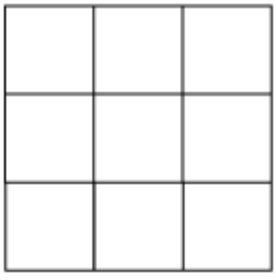
   (b) Determine a strategy (or a set of moves) that will guarantee that that player will win.



3. *Dominoes* is another game played on a rectangular board. In this game, players take turn placing dominoes (which are 1x2 rectangles) onto the board. The dominoes cannot overlap each other, and the first person who cannot play loses. A sample game on a 2x3 board where player 1 loses is shown below.

   (a) On a 3x3 game board (provided below) who should always win?

   (b) Why should that player always in? What strategy should they follow?



4. Play a few games of Sprouts with a friend. Start with a different number of dots each time. See if you can find a pattern in who should win. Why do we know this game will always end?

5. For each Nim game below:

   (a) Determine whether the game is a winning or losing game for player 1

   (b) If it is a winning game, find the winning move

   (c) Play each game with a friend and see if you can guarantee a win!

   I $*3 + *5$

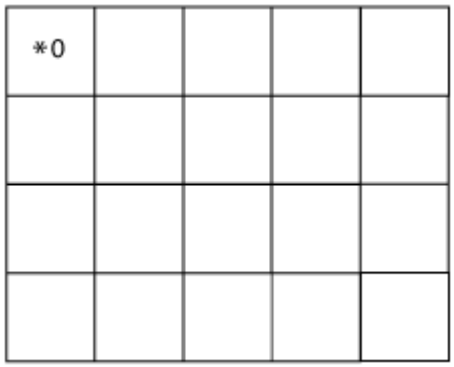
   II $*20 + *20$

   III $*3 + *3 + *4$

   IV $*1 + *1 + *1 + *1$

6. The Queens game is very similar to the Rooks game we played before. However, this time, the piece can also move diagonally up and to the left. Once again, the first player who cannot move loses. This means that if the Queen is in the top left corner of the board, then the player whose turn it is loses.

   On the board below, fill in the Nim values for each square. The top left square has been done for you.



Challenge Problems

1. Suppose we modified the game Nim by introducing a new rule. In addition to the chips in the piles on the board, there is now a bag containing x additional chips. On their turn, a player can either remove chips from any pile (a normal Nim move) or they can take chips from the bag and add them to any one pile.

   (a) Using the bag, can a player change a losing position in a normal Nim game into a winning game for them?

   (b) Why or why not?

2. In Chomp, if the starting board is strictly larger than a 1x1 rectangle (i.e. it has more than one square) who can always guarantee a win? (Think back to who won in the 2x2 and 3x2 cases from question 2 in the problem set). Prove that this player always has a winning strategy.