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Centre for Education in
Mathematics and Computing

Grade 6 Math Circles
Review Challenge - Solutions

November 24/25, 2015

Today marks the end of our Fall Math Circles journey. As such, this week does not include new topics but is a review challenge for you to tackle. You will need to use everything that you have learned over the past 7 weeks and more to complete this challenge. Should you get stuck, review the lessons from previous weeks to see if there are similar problems.

There are 7 'locks 'with each lock representing one week's Math Circles topics. The topics are in sequential order - so the lock 1 covers the topics from week 1. Solving these problems will give you the combinations to these 7 locks. The goal of this week is to find the combinations to each of the 7 locks.

Note: The questions on Estimation involved estimating the height/armspan of the instructor. This can only be done in person, and so an accurate estimate of height is included. The rest of the problem can be completed using the hints provided

Lock 1 - Algorithms

(a) You are given the following list of numbers: $66, 48, 18, 2, 60, 6, 22, 64, 86, 52$.

i. Find the 1st, 3rd, and 5th numbers that would be inserted using insertion sort. Add these numbers together.
Insertion sort works by taking the numbers from an unsorted list and adding them into an initially empty sorted list. The numbers are inserted in the order they appear in the unsorted list. This means that the 1st number is inserted is 66, the 2nd is 48, the 3rd 18 and so on.
This means that the 1st, 3rd, and 5th numbers are 68, 18, and 60 respectively. Their sum is 144

ii. Find the 1st, 3rd, and 5th numbers from the list created by sorting these numbers from smallest to largest. Add these numbers together.
When we have sorted the list from smallest to largest, the 1st number will be the smallest, the 2nd the second largest, and so on.
Our sorted list is: $2, 6, 18, 22, 48, 52, 60, 64, 66, 86$. This means the 1st, 3rd, and 5th numbers are 2, 18, and 48 respectively. Their sum is 68

iii. Take the larger of these numbers and subtract the smaller one. Divide the result by 2.
We first have $144 - 68 = 68$ Then, $\dfrac{76}{2} = 38$

iv. Call this number Q. Remember it. You will need it again later.
$Q = 38$

(b) You are given the following set of input and output values from a code-breaking machine.

Input	Output
1	4
3	18
5	40
9	108
10	130

i. Determine the algorithm used to convert these inputs into the given output.
Let x be our input. Then, our output is: $x^2 + 3x$

ii. Calculate the outputs for inputs of 2, 4, and 6
The outputs for 2, 4, and 6 are 10, 28, and 54 respectively.

iii. Multiply these values together and then divide by 2
We have: $10 \times 28 \times 54 = 15,120$ and then $\dfrac{15,120}{2} = 7,560$

iv. Call the result B. Remember it. You will need it again later.
$B = 7,560$

(c) Compute the following $(Q + B) - (\frac{Q}{B}) \times (B - Q) + (\frac{Q}{B} \times Q)$. Round your answer to the nearest whole number. This value is the combination for lock number 1.

$$\begin{aligned} Q + B &= 7,598 \\ \frac{Q}{B} &= 0.005026 \\ B - Q &= 7,522 \\ \frac{Q}{B} \times Q &= 0.191005291 \end{aligned}$$

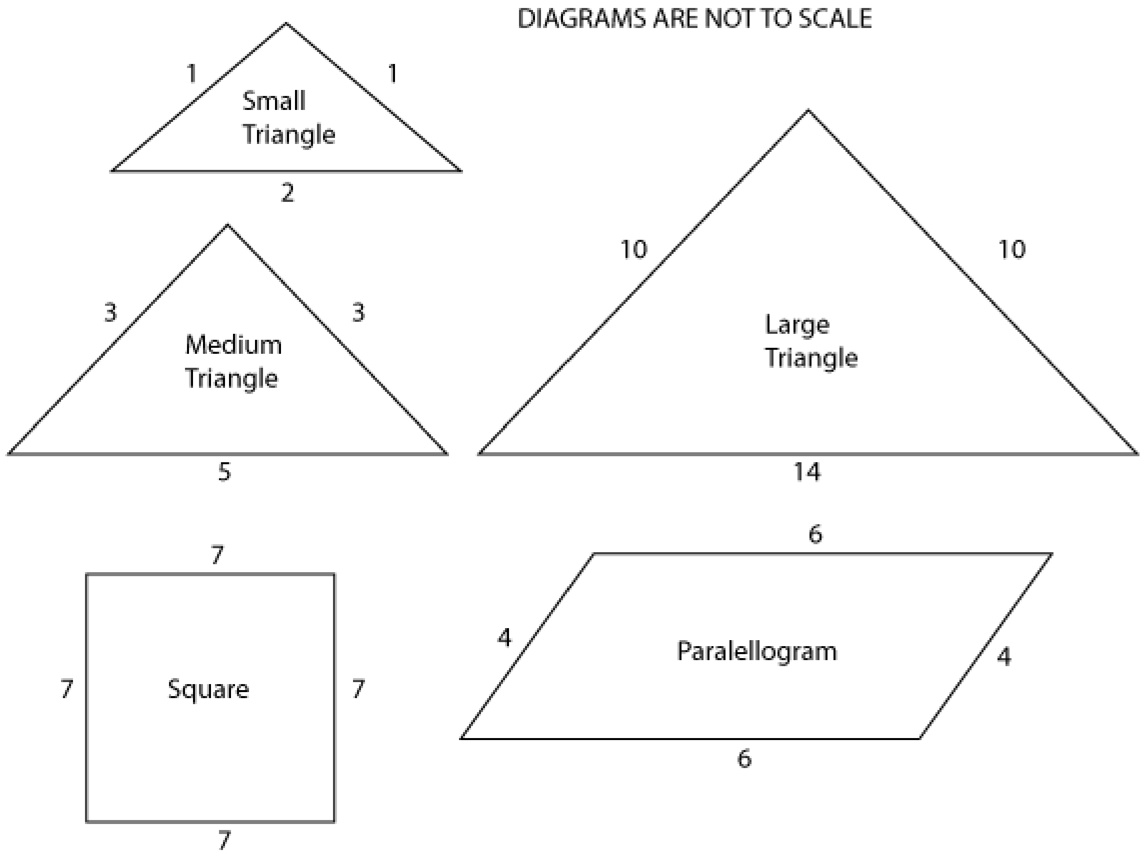
Combining these gives:

$$(Q + B) - (\frac{Q}{B}) \times (B - Q) + (\frac{Q}{B} \times Q) = 7,560.382011$$

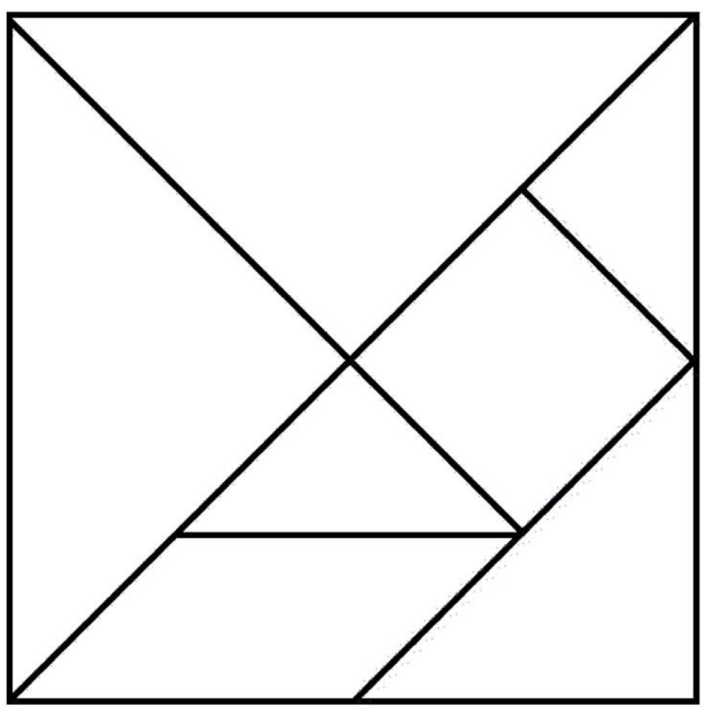
Rounding this to the nearest whole number means the combination is 7,560

Lock 2 - Tessellations

A set of tangrams is a set of 7 tiles. There is a square, a parallelogram, 2 small triangles, one medium triangle, and 2 large triangles. You will need a physical set of tangrams to complete these questions. A sample set is included at the end of this handout to print out to use. Each edge of the tangrams has been assigned a value. These values are shown below.



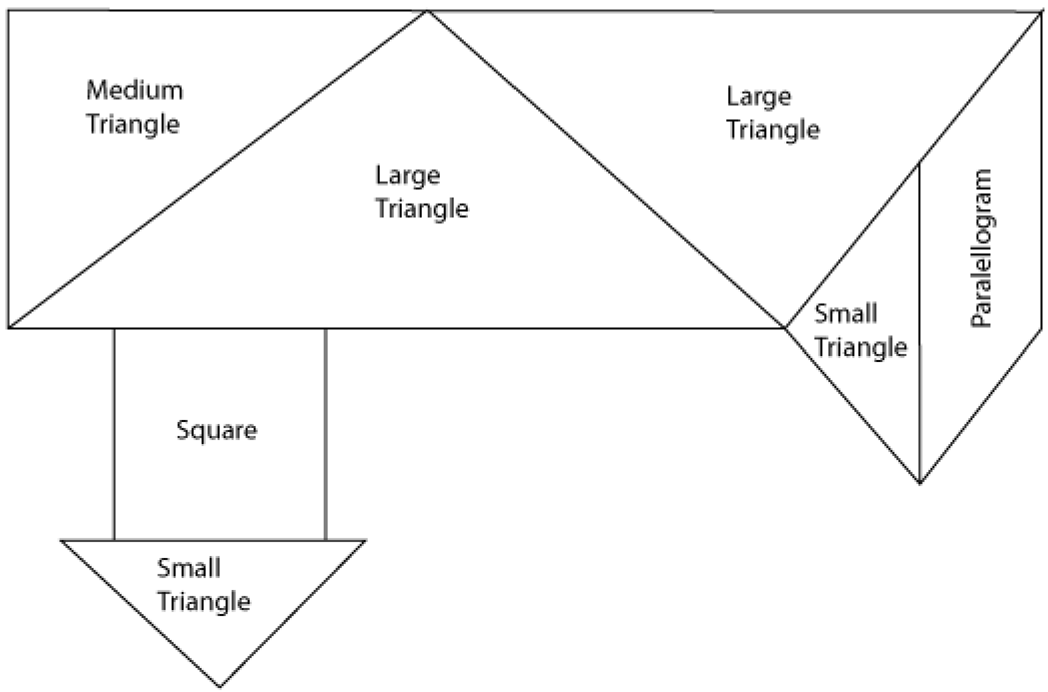
i. Build a square using all 7 tangrams. Multiply the values assigned to each outer edge to get a number.



The outer edges have values of: 14, 14, 2, 3, 3, and 6. Multiplying these together gives:

$$14 \times 14 \times 2 \times 3 \times 3 \times 6 = 21,168$$

ii. Build the shape found on the image below using all 7 pieces. Note that the image is not to scale. Multiply the values assigned to each outer edge to get a number. Partial edges count as a full edge, but only count them once.



The side lengths of the outer sides are (starting from the medium triangle):

$$3, 3, 14, 6, 4, 1, 14, 7, 2, 1, 1, 7$$

Remember, partial edges count once, and count as a whole edge.
Multiplying these together gives you the number: $4,148,928$

iii. Add the above two numbers. This is the combination for lock number 2.
We have

$$21,168 + 4,148,928 = 4,170,096$$

Lock 3 - Estimation

(a) Estimate my height in inches. *Use 74 inches*

(b) Estimate my arm span in centimetres.

Hint: An inch is 2.54cm

Hint: As in Leonardo Da Vinci's Vitruvian Man drawing, my arm span and my height could make up two side of a square

Using the hint that my arm span and height are two sides of a square, we can infer that my arm span is equal to my height. This means my arm span is 74 inches. We then convert this to centimetres. The exact number is

$$74 \times 2.4 = 177.6cm$$

We can also estimate this, as 2.54 is very close to 2.5. This means we need two and a half times 74. Half of 74 is 37, and two times 74 is 148. This means my arm span is approximately $185cm$.

(c) Multiply these two numbers together, and round to the nearest hundred. Add 3.14159. This is the combination for lock 3.
Depending on which numbers you use, you will obtain slightly different results. To illustrate the process we will use 74 inches as the height and $185cm$ as the arm span. Multiplying these together gives:

$$74 \times 185 = 13,690$$

Rounding to the nearest hundred and adding 3.14159 results in the combination being 13,703.14159

Lock 4 - Sequences

(a) The numbers $1, 4, 7, 10, ...$ form an arithmetic sequence.

i. Find the common difference between the terms of this sequence. Call this number X
The common difference is the difference between any two term. The difference between the first two terms is $4 - 1 = 3$. Therefore the common difference is 3. So, $X = 3$

ii. Find the next 3 terms in the sequence. Add these 3 terms together and multiply by 7. Call the result Y
To find the next term in the sequence, we just need to add the common difference to the previous term. This means the next 3 terms in the sequence are 13, 16, and 19. The sum of these terms is 48. Multiplying 48 by7 means that $Y = 48 \times 7 = 336$

(b) 27 and 18 are the 1st and 4th terms of an arithmetic sequence respectively.

i. Find the 2nd and 3rd terms of the sequence. Add these numbers and then multiply the sum by the common difference of the sequence. Call the result Z.
To find the 2nd and 3rd terms, we first need to find the common difference. first, we notice the following:
Let a be the first term of the sequence, and d be the common difference. Then:

$$\begin{aligned} \text{1st term} &= a \\ \text{2nd term} &= a + d \\ \text{3rd term} &= a + 2d \\ \text{4th term} &= a + 3d \end{aligned}$$

Since we know that the first term is 27, we know that $a = 27$. Also, since the 4th term is 18, we can solve for the common difference:

$$18 = a + 3d$$

Substitute a = 27

$$\begin{aligned} 18 &= 27 + 3d \\ -9 &= 3d \\ d &= -3 \end{aligned}$$

This means our common difference is -3. We can then find our 2nd and 3rd terms:

$$\begin{aligned} \text{2nd term} &= a + d = 27 - 3 = 24 \\ \text{3rd term} &= a + 2d = 27 - 6 = 21 \end{aligned}$$

Adding these numbers and multiplying by the common difference means that $Z = -135$

ii. Calculate the sum of the first 7 terms of the sequence. Call this sum V.
The first 7 terms of the sequence are $27, 24, 21, 18, 15, 12, 9$. Their sum is 126.

(c) Calculate $(X + Y) + (X + 2Z) + (X + 3V) - (Y \times Z) - \frac{V}{7}$. This is the combination to the 4th lock.
Using the values for X, Y, Z, and V found earlier, the combination is $45,795$

Lock 5 - Combinatorial Games

(a) Consider the the Nim game $*1 + *7 + *9 + *7 + *9 + *1$

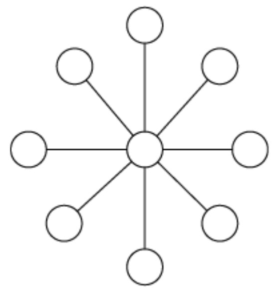
i. Which player (1 or 2) will win this game?
Player 2 will win as we can pair off all of the piles

ii. Assuming the players follow a copycat strategy, determine the number of chips the winning player will take.
In a copycat strategy, on each turn player 2 takes exactly the same number of chips as player 1. This means they take the same number of chips total. This results in each player taking half of the chips. This is $1 + 7 + 9 = 17$

iii. Multiply the number of chips taken by the winning player by the winning player's number. Call the result S.
$S = 2 * 17 = 34$

(b) Consider the Nim game $*9 + *3 + *3$. Determine the number of the winning player (1 or 2). Call this number H.
After pairing off the two piles of 3, we are left with one pile left. This means that player 1 will win. therefore $H = 1$

(c) In the Sun game, two players take turns placing discs numbered 1 to 9 in the circles on the board (shown below). Each number can only be used once. The object of the game is to be the first to place a disc so that the sum of the numbers along a line through the centre circle is 15.



What number, when placed in the middle circle, can guarantee player 1 a win? Multiply this number by 11 and call this N.
The right number to play in the middle circle is 5. If 5 is in the middle circle, then each line of 3 circles has a 5 in it. Since we want the sum on each line to be 15, the other two numbers must add up to 10. No matter what number player 2 plays, player 1 can play along the same line with the number that will make the sum of those two add up to 10. We would only run into an issue if player 2 played 5, because then to add up to 10 we would need to duplicate 5. But since the 5 has already been played, player 2 cannot play 5.
This means $n = 5 \times 11 = 55$

(d) In the Nim game $*1 + *7 + *1 + *9$ Player 1 can guarantee a win. Number the piles from left to right (so the first $*1$ is pile 1, and the $*9$ pile is pile 4). Divide the pile number by the number of chips removed from that pile. Call the result A.
Player 1 can guarantee a win if, after their turn, we can pair up all of the Nim piles. Initially, the two piles of 1 can be paired up - so we don't need to touch them. We would like to make the piles of 7 and 9 have the same number of chips. This means taking 2 chips from the pile of 9. The pile of 9 is pile number 4. This means that $A = \frac{4}{2} = 2$

(e) Arrange the numbers you have to form the word NASH. (e.g. if you had A = 10 and B = 24, then ABBA would be 10242410). This number is the key to the 5th lock.

$$NASH = 554341$$

Lock 6 - Logic

Use the grids at the end of this handout for questions (a) and (b)

(a) Expensive Homes: Three members of a secret society live in homes worth more than $ 1 million. From the following clues, determine each person's last name, what street they live on, and how much their home is worth.

i. Agent K's home is worth more than the home of the agent whose last name is Holloway
ii. The Elm Tree Road agent owns the least valued home
iii. Agent A does not live on Treelined Boulevard, but in the most valuable home.
iv. Holt is not the last name of the Wattle Grove resident

Agent	Last Name	House Value	Street
Agent A	Meltosia	$ 1,499,000	Wattle Grove
Agent G	Holloway	$ 1,050,000	Elm Tree
Agent K	Holt	$ 1,278,500	Treelines

(b) Jigsaws: The National Jigsaw Puzzle Society has recently discovered a number of codes hidden in some of their jigsaw puzzles. Each puzzle featured a different theme and was produced by a different company. None of the puzzles were released in the same year. Determine the number of pieces in each puzzle, what the theme was, who produced the puzzle, and what year it was released. Use the following clues that the National Society has left for you:

i. The jigsaw puzzle with 1050 pieces doesn't have the railroad theme
ii. The puzzle with 1300 pieces wasn't released in 1980
iii. The puzzle made by Eduka has 500 more pieces than the puzzle released in 1991
iv. The puzzle with the orchard theme is either the jigsaw puzzle made by Hasbro or the puzzle released in 1980
v. The jigsaw puzzle released in 1984 is either the puzzle made by Ceaco or the puzzle featuring cows
vi. 3 of the puzzles consist of the puzzle made by Eduka, the jigsaw puzzle with 1050 pieces, and the puzzle with 800 pieces
vii. Of the jigsaw puzzle with 800 pieces and the jigsaw puzzle made by Hasbro, one has the orchard theme and the other has the plantation theme
viii. The jigsaw puzzle made by Buffalo has 750 more pieces than the jigsaw puzzle released in 1990
ix. The jigsaw puzzle with the orchard theme has 250 fewer pieces than the puzzle depicting cows
x. Hasbro did not make a puzzle of 550 pieces
xi. The puzzle with the plantation theme had more pieces than the puzzle featuring cows

Pieces	Company	Theme	Year
300	Hasbro	Orchard	1990
550	Masterpiece	Cows	1984
800	Ceaco	Plantation	91
1,050	Buffalo	Dogs	1980
1,300	Eduka	Railroad	1996

(c) Solve the following 4x4 KenKen Puzzle

1 - 3	3 - 4	1	2 2
2	2 ÷ 1	1 - 3	4
3 - 1	2	24 x 4	4 + 3
4	3	2	1

(d) Divide Agent K's house value by the number in the bottom left corner of the KenKen grid. Multiply the number of pieces in the railroad themed puzzle by the year the Hasbro puzzle was released. Add these two numbers. This is the combination for lock number 6.

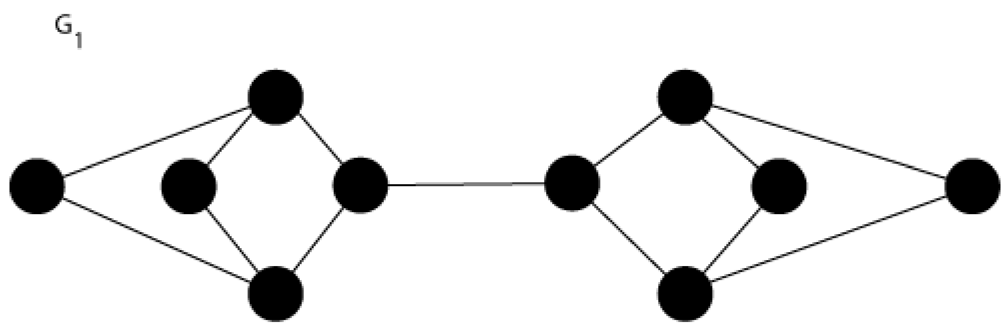
Clue	Number
Agent K's House Value	1,278,500
Bottom Left Corner of KenKen	4
# of pieces in Railroad Puzzle	1,300
Hasbro Year	1990

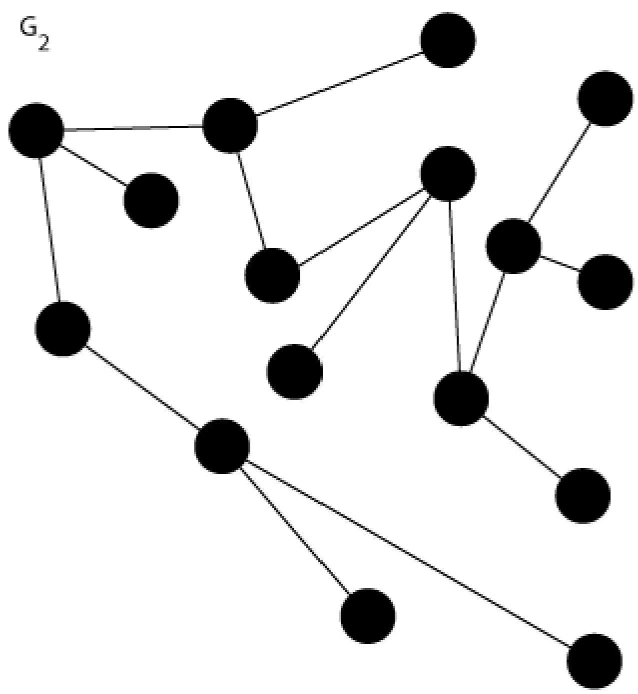
Using these values we know the combinations is:

$$\frac{1,278,500}{4} + (1300 \times 1990) = 2,906,625$$

Lock 7 - Graph Theory

(a) Count the number of edges and vertices in each of the two graphs (G_1 & G_2) below. Add all of those numbers. Call the result E.
There are 10 vertices and 13 edges in G_1. There are 16 vertices and 15 edges in G_2. Adding these together means that $E = 54$

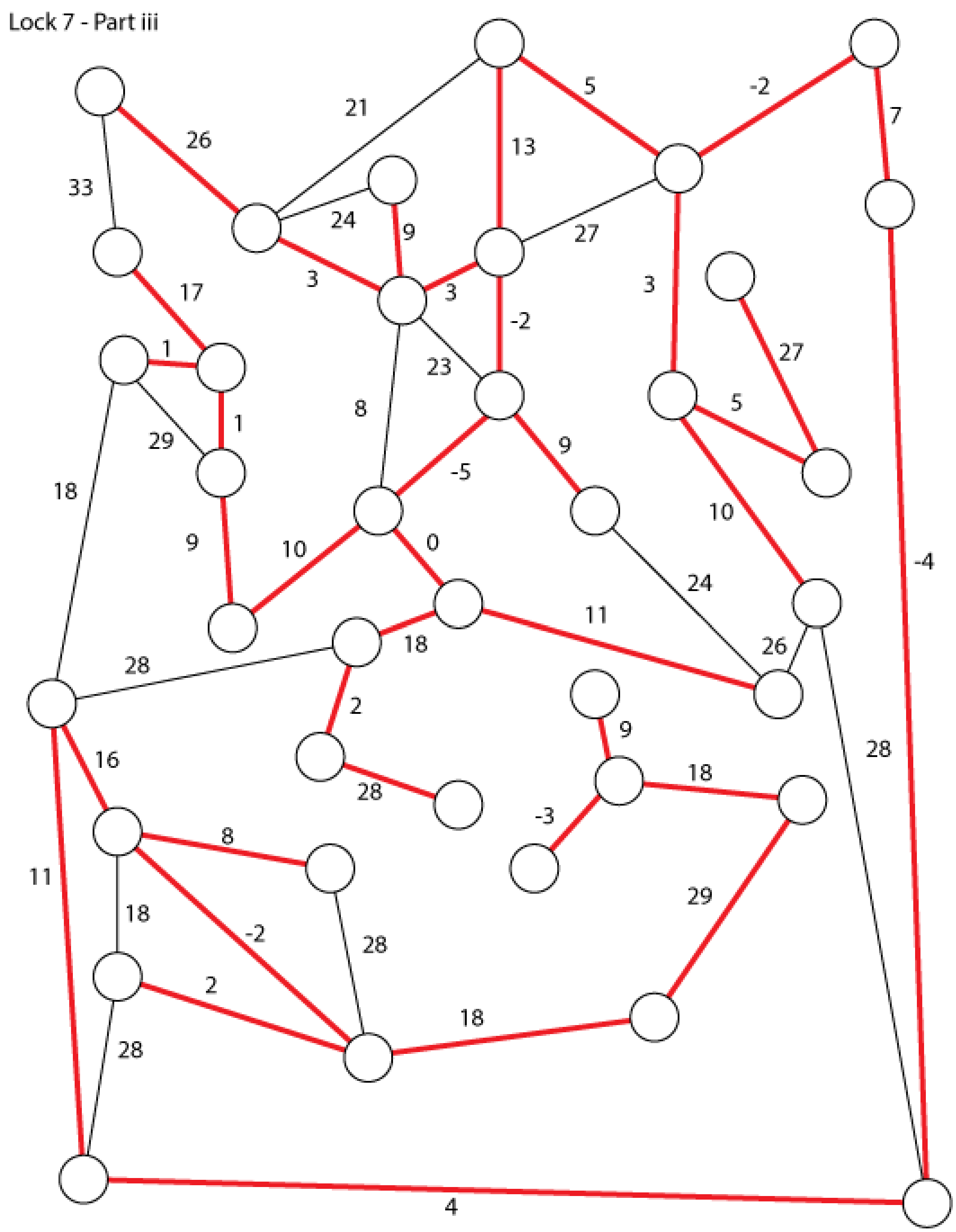




(b) Determine the total number of distinct cycles in G_1. Remember that two cycles using the same vertices that simply start in different places are NOT distinct. Call this

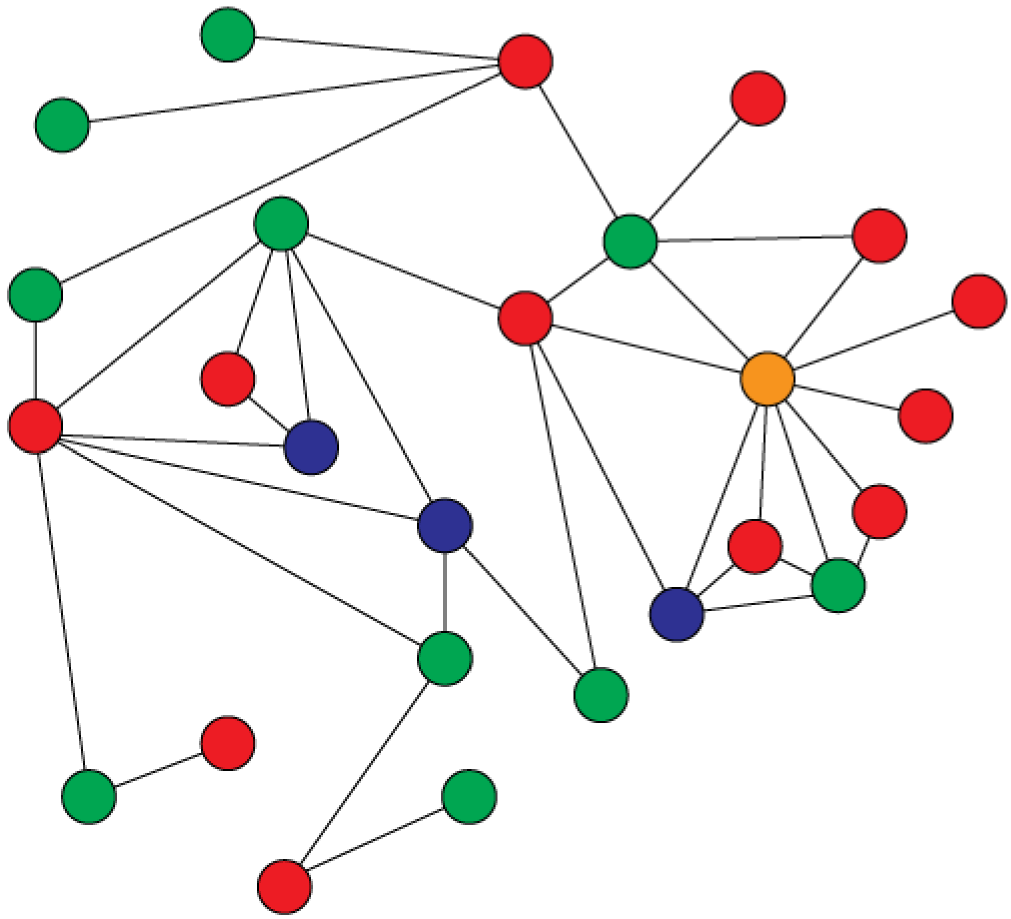
number S.
There are 6 distinct cycles in G_1. There are 3 cycles in each group of 5 vertices. Therefore $S = 6$

(c) Find a minimum spanning tree in the below graph (you can start with any vertex you want). Call the weight of this spanning tree G.



The weight of the MST (the red edges) is 319

(d) Colour the graph below with the fewest colours possible. Call this number D.



This graph needs 4 colours. Therefore $D = 4$

(e) Arrange the numbers you have to form the word EDGES. (e.g. if you had A = 10 and B = 24, then ABBA would be 10242410). This number, divided by 2 plus the number of bridges in Königsberg is the key to the 7th lock. We have that:

$$E = 54S \qquad = 6G = 319D \qquad = 4$$

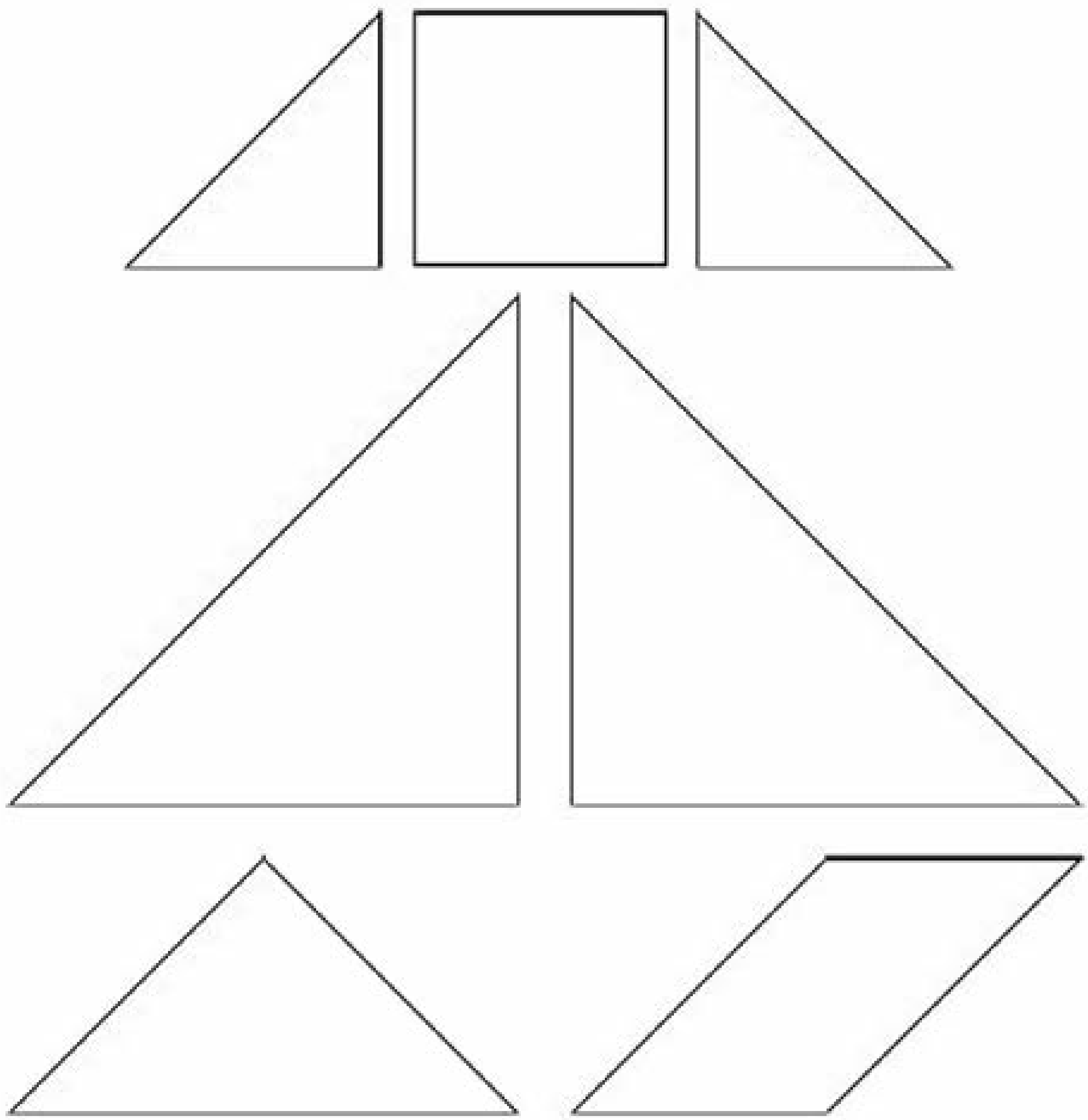
Therefore

$$EDGES = 544319546$$

There were 7 bridges in the Königsberg problem, so our final combinations is

$$\frac{544319546}{2} - 7 = 272,159,780$$

Tangram Set - For Printing



Expensive Homes Logic Grid

		Last Name			Value			Street		
		Holloway	Holt	Meltosia	$1,050,000	$1,278,500	$1,499,000	Elm Tree	Treelines	Wattle Grove
Agent	Agent A									
	Agent G									
	Agent K									
Street	Elm Tree									
	Treelines									
	Wattle Grove									
Value	$1,050,000									
	$1,278,500									
	$1,499,000									

Jigsaws Logic Grid

		Company					Theme					Year Released				
		Buffalo	Ceaco	Eduka	Hasbro	Masterpiece	Cows	Dogs	Orchard	Plantation	Railroad	1980	1984	1990	1991	1996
Piece Count	300															
	550															
	800															
	1050															
	1300															
Year Released	1980															
	1984															
	1990															
	1991															
	1996															
Theme	Cows															
	Dogs															
	Orchard															
	Plantation															
	Railroad															