



Grade 7/8 Math Circles

November 10/11, 2015

Logic

Truth

Logic is the study of arguments, their validity, and their truth values. What do you know about logic? Puzzles, such as SuDoKu and algebra problems use logic. Ideas are called logical or illogical. Paradoxes seem to follow logically but give illogical results. How does logic operate in all of these activities? To use logic, we need the ability to write down and analyse sentences that deal with information. We need to be able to assess some basic truths before discovering the truth or falsity of more complex statements. To do all of this, it is convenient to learn about arguments, logical operators, truth tables, and Minecraft.

Once we do all of that, we will have some tools at our disposal to understand sentences and problems logically and produce solutions that are logical — or, in other words, follow from our assumptions. As mathematicians, we do this all the time. We take word problems, the numbers they provide, and make an argument. We have used the word problem as our assumption and the calculations in a solution are our argument, and our answer is a conclusion.

Making Arguments

Every argument has premises and a conclusion. If the conclusion follows from the premises, then the argument is *valid*. If the premises are true and the argument is valid, then the argument is *sound*. We will see simple arguments today that have two premises and one conclusion. Each of these pieces can also be called a statement. A *statement* is a sentence that is either true or false or whose truth we can determine.

Some statements can be written as if/then statements. For example, if I eat lots of candy, then I will be sick. This type of statement is like a small or simple argument that has a single premise and a conclusion of its own. My example is reasonable, so we will accept it as true. Now here is our first argument:

Premise 1: If I eat lots of candy, I will be sick.

Premise 2: I just ate 1000 Twizzlers.

Conclusion: I will be sick.

I probably did not need to write the conclusion, you could have guessed it from the premises. That is because the form of this argument: If A then B. A. Therefore B; is a valid form that must give a true conclusion if the premises are true. Consider this argument:

Premise 1: If I eat lots of candy, I will be sick.

Premise 2: I am sick.

Conclusion: I ate way too much candy earlier.

Is the conclusion true? No. There are many reasons for being sick and candy is not the only cause. Therefore we cannot draw this conclusion. The form of this argument is: If A then B. B. therefore A; and it is invalid. There will always be a counter-example in an argument with this form. Whereas, if the form is valid, then there will not be any counter-examples to the argument, as long as it had true premises.

Paradoxes

Logical paradoxes are often arguments that seem to give a conclusion that follows logically from the premises but still contradicts the premises or some other truth. For example, determine whether the following sentence is true or false:

This sentence is false.

If it is false, then the sentence is actually true. But if the sentence is true, then it is not false, so the sentence is false. Here is the argument written with premises:

This sentence is false.

If that sentence is false, then it is true.

Therefore, that sentence is true.

The conclusion is in direct contradiction to the premise, yet the argument is in valid form! What gives? Often paradoxes have solutions. One solution to this paradox is to avoid self-referencing in a sentence. A sentence cannot make a claim about its own truth; its truth must be decided in advance.

Exercise: Think of some other paradoxes you have seen. Can you identify their premises and conclusion?

Logical Operators

Recall what *order of operations* means. It is a description of the order in which we should perform operations, like adding and multiplying. Adding is called an operator because it operates on two inputs (just two numbers). There are other operators besides adding, subtracting, multiplying, and dividing. There are also operators that operate on items other than numbers. For example, in logic we have several operators that take either true or false as input. It may make more sense in certain situations to represent true with either on or a 1, and false with either off or a 0. Here are four logical operators (and a recent Google Doodle, for George Boole's 200th birthday, utilizing them <https://g.co/doodle/cjk8vt>):

NOT takes one input and produces the opposite output. NOT is represented by the symbol ' \neg '. For example, $\neg True \rightarrow False$ and $\neg False \rightarrow True$.

OR takes two inputs and produces true if true is at least one of the inputs and false if and only if both inputs were false. OR is represented by the symbol ' \vee '. For example, $True \vee False \rightarrow True$.

AND takes two inputs and produces true if and only if both inputs were true and false if at least one of the inputs was false. AND is represented by the symbol ' \wedge '. For example, $True \wedge False \rightarrow False$.

XOR takes two inputs and produces true if and only if only both inputs are different and false if and only if both inputs are the same. XOR is represented by the symbol ' $\underline{\vee}$ '. For example, $True \underline{\vee} True \rightarrow False$.

We can immediately combine the last three operators with NOT to create NOR, NAND, and XNOR. Put simply, they produce the opposite outputs from OR, AND, and XOR, respectfully.

NOR takes two inputs and produces false if true is at least one of the inputs and true if and only if both inputs were false. For example, $True \vee False \rightarrow False$.

NAND takes two inputs and produces true if and only if both inputs were false and true if at least one of the inputs was false. For example, $True \wedge False \rightarrow True$.

XNOR takes two inputs and produces true if and only if only both inputs are the same and false if and only if both inputs are different. For example, $True \underline{\vee} True \rightarrow True$.

Exercise: What happens if I input true into two NOT operators that have been combined? In other words, what is the value of $\neg\neg True$? $\neg\neg True = \neg False = True$.

Truth Tables

Truth tables are summaries of logical operators. The INPUT columns of our truth tables will, together, list all possible combinations for input into an operator. For our operators with two inputs, this is relatively easy. Since we only have two types of input values — true or false — we only have four possible combinations of inputs. For the our logical operators today, the order of the inputs does not matter.

The truth table for NOT:

INPUT	OUTPUT
True	False
False	True

The truth table for OR:

INPUT1	INPUT2	OUTPUT
True	True	True
True	False	True
False	True	True
False	False	False

Now complete the truth table for AND:

INPUT1	INPUT2	OUTPUT
True	True	True
True	False	False
False	True	False
False	False	False

The truth table for XOR:

INPUT1	INPUT2	OUTPUT
True	True	False
True	False	True
False	True	True
False	False	False

Complete the truth table for NOR:

INPUT1	INPUT2	OUTPUT
True	True	False
True	False	False
False	True	False
False	False	True

Now complete the truth table for NAND:

INPUT1	INPUT2	OUTPUT
True	True	False
True	False	True
False	True	True
False	False	True

Now complete the truth table for XNOR:

INPUT1	INPUT2	OUTPUT
True	True	True
True	False	False
False	True	False
False	False	True

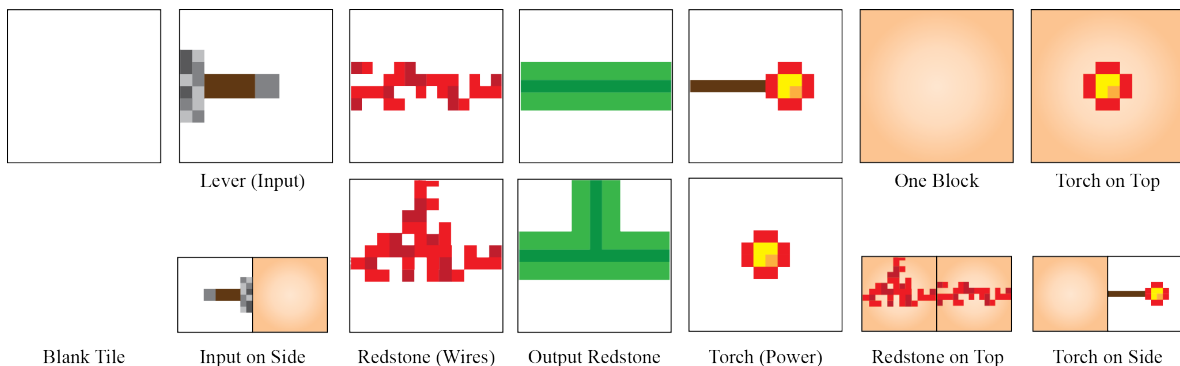
Minecraft and Redstone

Minecraft is an open world video game where the player is tasked with surviving and given the ability to completely alter his/her world. There are several mechanisms in Minecraft for performing different abilities. Redstone is one of these mechanisms that allows the player to make everything from automatic door openers to computers, all within the game. Unlike the real world, where door openers and computers can be made very tiny, the blocks in Minecraft are all that you are given so some contraptions become very large. Making them as small as possible or in different shapes that can be hidden well will make your contraptions look better. Our contraptions today will be ugly because we only care about learning the logic circuitry behind redstone.

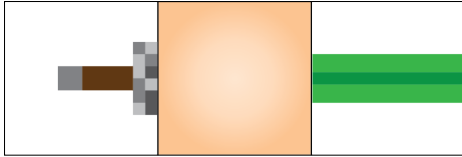
Here are a few rules we will need to know about using redstone in Minecraft:

- redstone acts as wires in Minecraft, carrying a signal a total of 15 blocks away
- the game measures time in ticks; one tick is 1/20 of a second and redstone has a separate tick that is twice as long, one redstone tick is 1/10 of a second
- repeaters are items in the game that can make your redstone signal go further; they also delay your signal by 1-4 redstone ticks depending on their setting
- redstone torches act as a power source, so do various input methods
- there are many inputs such levers, buttons, pressure plates, light sensors, and trip wires
- the whole idea behind logic gates in Minecraft is to combine them to solve a problem; think of the circuits as logic puzzles
- a block can be powered by feeding a signal to it; signals do not easily go up and down, there are tricks for these situations

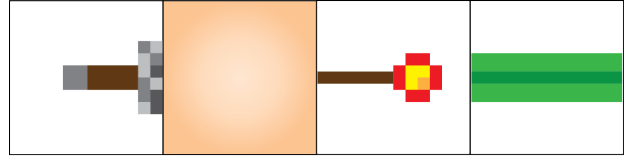
Legend of Schematic Symbols



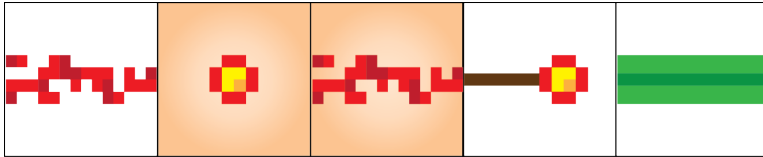
What follows are a list of logic gate schematics for Minecraft. Each of the inputs can be true or false and the green output wiring is going to send out either a true or false signal depending on the logic gate and input values.



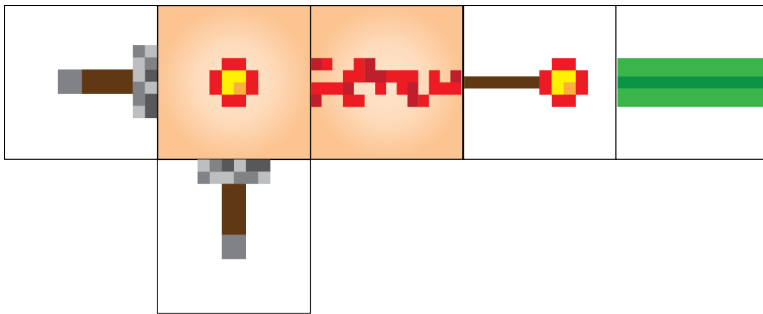
Simple Input/Output Gate (on/off switch)



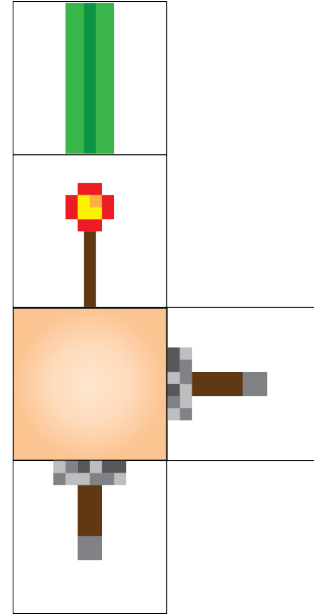
NOT Gate (reverses on/off switch)



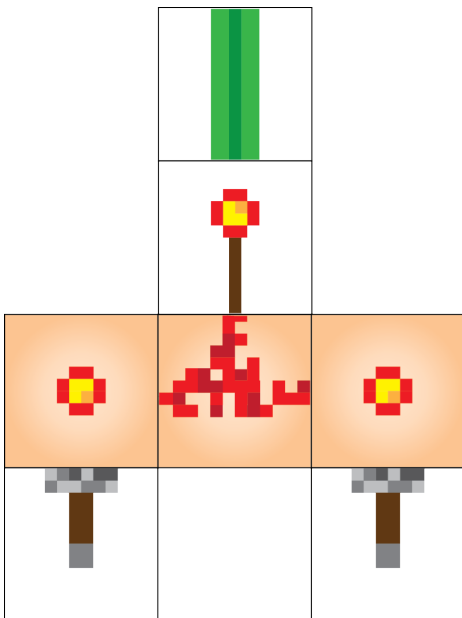
Traditional Repeater (a one block repeater exists in game)



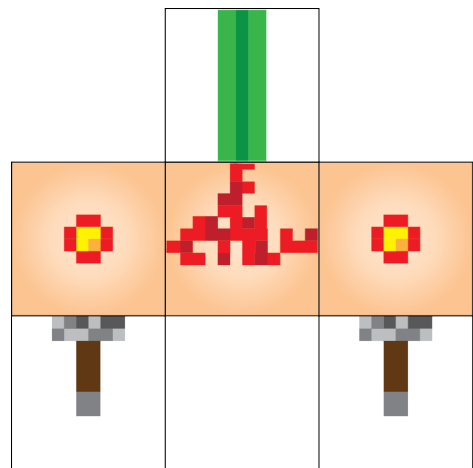
OR Gate (notice the similarities to the traditional repeater)



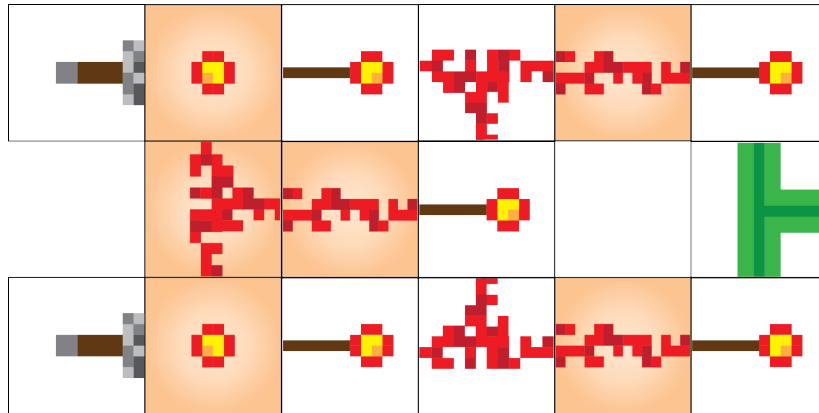
NOR Gate (notice the similarities to the OR Gate)



AND Gate (NOT + NOT + AND)



NAND Gate (NOT + AND)



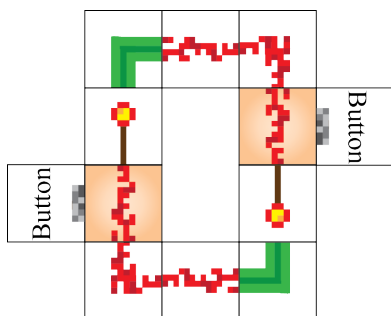
XOR Gate (exclusive OR)

Memory

Redstone mechanics are simple but sophisticated enough to build actual computers in the game. To build a computer, we need memory but how does one build such a thing in Minecraft? As it turns out, a simple combination of logic gates that we have already seen will do the trick.

A computer is a logic machine and, at the most basic level, can only understand two states: either true or false. Therefore, there are only two states a memory cell in our computer needs: either it needs to store the value true or false. If we use 0 for false and 1 for true, then we have a new number system called binary at our disposal. We do not need the 10 numbers we are accustomed to to count, we will use only the numbers 0 and 1. Here, I will count to 10 in binary, starting at 0:

0	0	3	11	6	110	9	1001
1	1	4	100	7	111	10	1010
2	10	5	101	8	1000		



On the left is our memory circuit. Pretend the bottom green output is off and the top output is on. Pressing the left button turns off the left NOR gate which turns on the right NOR gate. Now the bottom and top outputs are reversed and it will stay this way until we switch them again. If we connect more Redstone to it, we can have either a true or false output depending on its current state. All it took was two NOR gates.

The output from a memory cell can be true or false, or in other words, 1 or 0. If we had four of these cells, we could store any number between 0 and 15 thanks to binary.

Exercises

1. Complete each argument by writing a conclusion that follows logically.
 - If Marley or Sandra run in the race today, then I will lose.
Besides myself, Dennis, Felicia, Sandra, and Martin all ran in the race today.
So in the race today, I **lost**.
 - If I buy a phone and a computer, I will not be able to buy a smart watch.
I had to buy both for school, so I **cannot buy a smart watch**.
 - Either I can take the money or leave it.
I didn't leave it, so I **took the money**.
 - If I play Minecraft on Sunday, then I will miss my TV show.
My Sunday TV show was awesome!
Therefore: I **did not play Minecraft**.
2. Recall that an argument can be valid but not sound. Soundness depends on whether or not the premises are actually true. An argument can follow logically from untrue premises. So you can make logical arguments that are not true. Here are three arguments: one is **invalid and not sound**, one is **valid but not sound**, and the other is **valid and sound**. Determine which argument is which out of the **three options**.
 - All Canadian residents over 18 have the right to vote in an election.
You are in grade 8 and are a Canadian resident.
Therefore, you have the right to vote.
Not valid and not sound.
 - All whole numbers greater than 1 have a prime factorization.
1960 is a whole number.
Therefore, it has a prime factorization and it is $2 \times 2 \times 2 \times 5 \times 7 \times 7$.
Valid and sound.
 - All animals with wings can fly.
Penguins have wings.
Therefore, penguins can fly.
Valid but not sound.

3. Arrange the following sentences to make a valid argument. Is the argument sound?
 Technically, you may arrange the premises and conclusion in any order you want. The order I have chosen makes the most sense with the language I have used and puts the argument in proper logical form (If A then B. A. Therefore, B. Replace A and B with statements.)

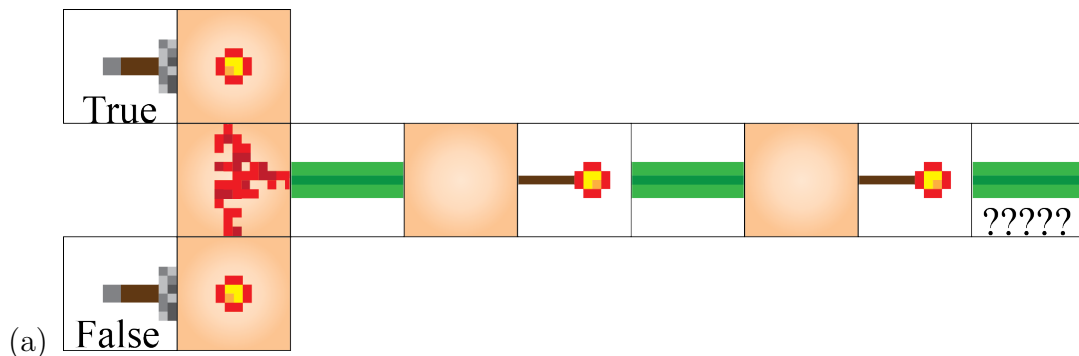
(a) Therefore, they cannot fly. — Monkeys have no wings. — All creatures without wings cannot fly.

All creatures without wings cannot fly. Monkeys have no wings. Therefore, they cannot fly.

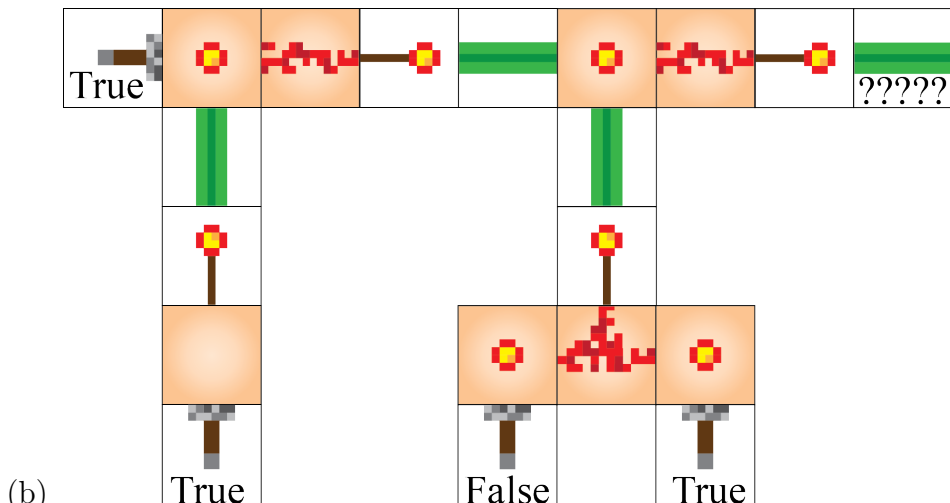
(b) $(1 + 4)^2$. — Evaluate brackets before exponents. — $= 5^2$.

Evaluate brackets before exponents. $(1 + 4)^2 = 5^2$.

4. Determine what value the final output has: true or false?



False.



True.

5. SuDoKu is a puzzle-game about filling squares with the numbers 1 through 9. Every column, row, and 3×3 box must contain the numbers 1 to 9 with no repeats. There

are numbers already given as hints. Try out the following two puzzles for yourself: [No solutions for these puzzles are provided.](#)

		8	6	5		2		
		6		3	8			5
2		7		1	9	6		
		2		6	4			
7	1						2	6
			1	9		8		
		5	3	7		1		2
3			9	4		7		
		9		2	1	5		

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5		9	2					4
	6		8				5	
			4	5		6		
					9	4		1
	8						7	
7		3	1					
		5		2	3			
	1				8		3	
8					4	7		5

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6. Recall that logical operators work in a similar way to how $+$, $-$, \times , \div are used. Find out what these statements equal (it will either be **true** or **false**). You may want to review the symbols being used and what they mean.

(a) $(True \wedge False) \vee True = \text{True}$

(b) $(\neg True) \wedge (False \vee False \vee False \vee True \vee False) = \text{False}$

(c) $\neg((False \vee True) \vee True) = \text{True}$

(d) $(\neg(True \vee True)) \wedge False = \text{False}$

7. I have three boxes that I know are labelled incorrectly. Their labels are “lego”, “megablocks”, and “lego and megablocks”. If you can only open one of them, which box will you open to guarantee that you know how to label the boxes correctly?

[Open the lego and megablocks box because you know every box is labelled incorrectly so if lego is in that box, you know the lego label goes on it, the megablocks label goes on the lego box and the lego and megablocks label goes on the megablocks box.](#)

8. You have two jugs, one is three litres and the other five; how can you use these to measure four litres? Note: You need four litres exactly and each jug has no markings for any measurement other than three or five litres.

[Pour the 5 litre jug full and empty 3 litres into the 3L-jug. The 5L-jug now contains 2L. Empty the 3L-jug and pour the 2L into it. Fill the 5L jug. Now the 3L jug has 1L](#)

of empty space, so fill the 3L-jug with 1L from the 5L-jug. Now the 5L-jug contains 4L.

9. * This logic problem became famous last year and made its rounds on Facebook. See if you can solve it.

Cheryl's Birthday is one of the 10 possible dates:

May 15 May 16 May 19
June 17 June 18
July 14 July 16
August 14 August 15 August 17

Cheryl tells the month to Albert and the day to Bertrand.

Albert says, "I don't know the birthday, but I know Bertrand doesn't know either."

Bertrand then says, "I didn't know at first, but now I do know."

Albert then says, "Now I also know Cheryl's birthday."

When is Cheryl's birthday?

For a full solution, watch this video by Brady Haran <https://youtu.be/emIMj8cCL5E>. To check your own work, Cheryl's birthday is July 16 BUT not everyone is guaranteed to get this solution. Watch this video after watching the first to see why: <https://youtu.be/Mj-YngjjpJg>.

10. * Your math teacher gives you 12 coins and tells you that one of them is fake. The fake coin is almost perfect, the only difference between it and the other 11 coins is that it is slightly lighter. Your teacher also gives you a simple scale that can compare only two coins or two groups of coins together. It will only tell you which side is heavier. You are not allowed to do anything but measure the coins on the scale. What is a strategy that will always find the fake in the smallest number measurements on the scale? How many steps would it take your strategy to find the fake coin in the worst case scenario? Always split the coin pile in half. Split the coins into two piles of 6, measure them, the lighter pile contains the fake. Split that pile into two piles of three coins each, measure them, the lighter pile contains the fake. Do this process again, but with 1 coin and 2 coins. The lighter pile contains the fake. If the 1 pile was lighter, it is your fake, if not then measure the last two coins. The worst case scenario is this 2 coin pile being lighter and therefore takes a total of four measurements. However, it can find the coin in as low as 3 measurements.