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Grade 7/8 Math Circles

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Knot Theory

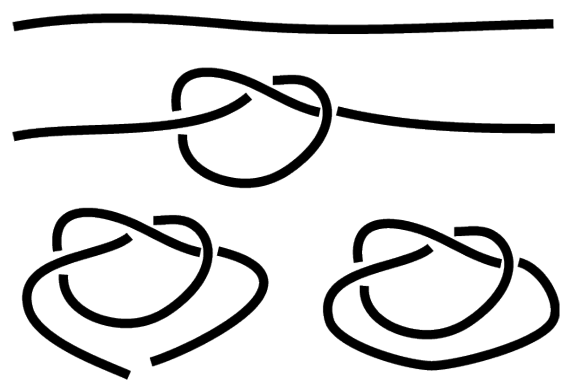
What are mathematical knots?

Mathematical knots are not the same ones you use to tie your shoes. In mathematics, knots are closed loops (they do not have ends) like a circle. In fact, a circle is a knot, known as an unknot or a trivial knot because it is so simple. A knot that is not trivial has some sort of folding in space such that no amount of steps can undo the knot into an unknot. Find a piece of yarn, string, or a pipe-cleaner and follow these steps to make your first knot, the trefoil knot:

1. Lay the string out flat and horizontal.

2. Tie a simple overhand knot in the string but do not tighten it.

3. Connect both ends of the string together.

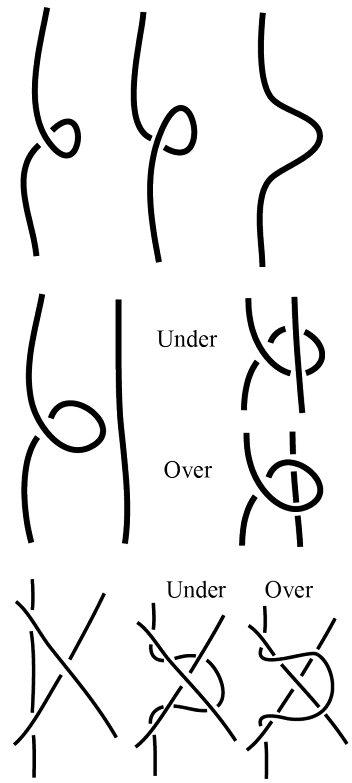


Before you begin manipulating this knot, we should discuss some rules that need to followed. There are moves that you can and cannot do to move pieces in a knot. The most important criteria is what you **cannot** do.

- A string cannot pass through string (like a ghost through a wall).

- A string cannot be cut and glued back together (we can do this to make the knot but not while manipulating it).

If you reflect on why we cannot perform these actions, you will come upon a list of actions we can perform on knots that will not break the above two rules. Almost a century ago, a mathematician did just this. The allowable actions are called Reidemeister moves, named after the German mathematician Kurt Reidemeister.



I. Twist or untwist a loop in any direction.

II. A loop can move over or under a string.

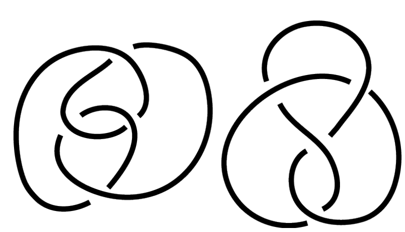
III. A string can move over or under a crossing.

Now take the trefoil knot you made and play around with it. As long as you do not cut your knot, you will be following the Reidemeister moves. See if you can find a way to ‘undo’ the knot using only Reidemeister moves. Is such a feat possible?

In fact, no amount of tangling or untangling will make the trefoil knot become the unknot. Therefore, the trefoil knot and the unknot are two distinct knots — it is possible to identify them apart from each other. It might seem obvious after playing with the trefoil knot that it is in fact different from the unknot but how do I *know* that there is *no* set of moves you or I could perform to reveal it to be just the unknot? Think about all the ways you can make the trefoil knot more complicated. How am I certain that five more minutes of playing with it will not transform the trefoil into an unknot?

Minimal Crossing Number

As you may already see, one knot can look very different depending on how it is twisted or deformed. A single ‘configuration’ of a knot is called a projection. For example, on the right are two projections of the figure-8 knot — that is, they look different but are actually the same knot.

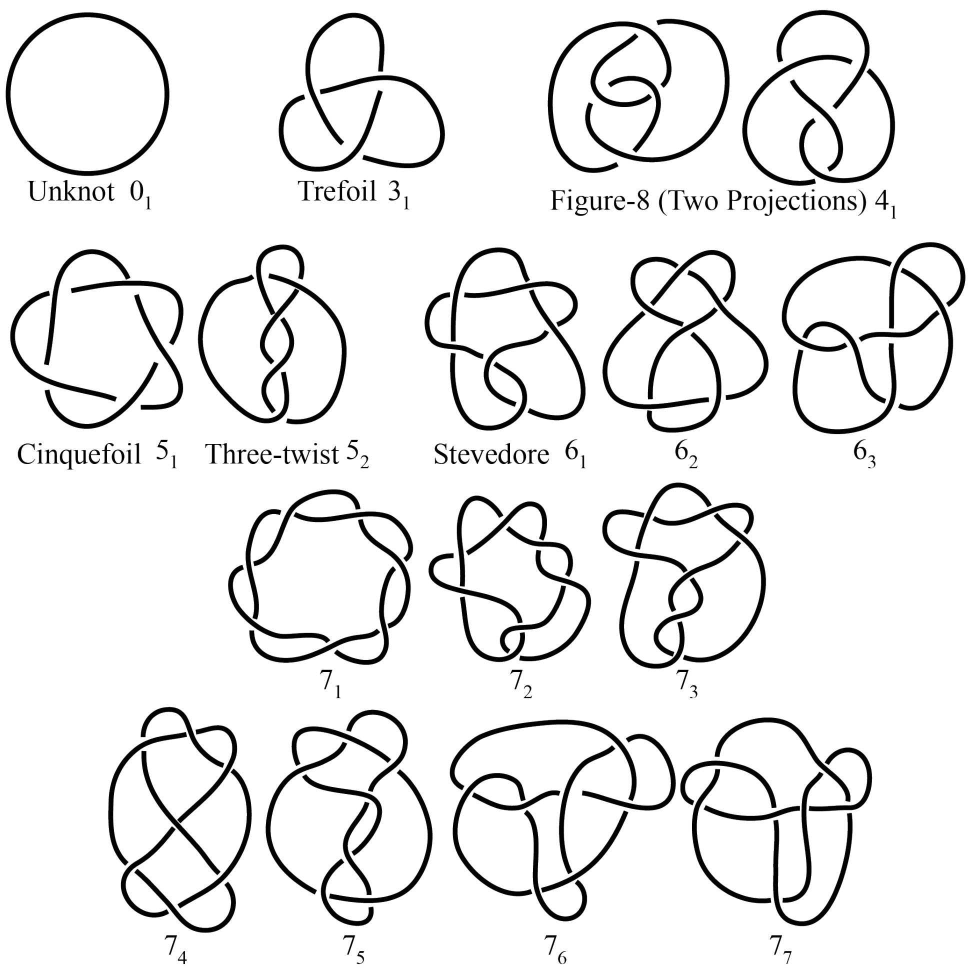


Two knots are the same if there is a sequence of Reidemeister moves to turn one knot into the other.

Both of these figures have the same number of crossings. A *crossing* is where a piece of the knot goes over and underneath itself. Notice that they do not need to have only four crossings. Using any of the Reidemeister moves, we can transform the knot into a projection with more crossings. Therefore, knot theorists will often refer to knots by their *minimal crossing number*, or in other words, the smallest number of crossings out of all projections of a knot. This number is actually very difficult to find. How do we *know* that we cannot perform some set of moves on a figure-8 projection to transform it into a trefoil knot?

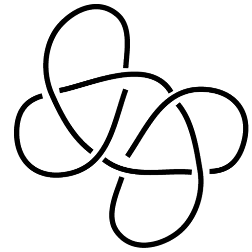
Prime Knots

Prime knots are like building blocks of all knots. There are many knots and some can be broken down into other, simpler knots. Prime knots cannot be separated into other knots. Watch this video on prime knots by Brady Haran: `https://youtu.be/M-i9v9VfCrs`.

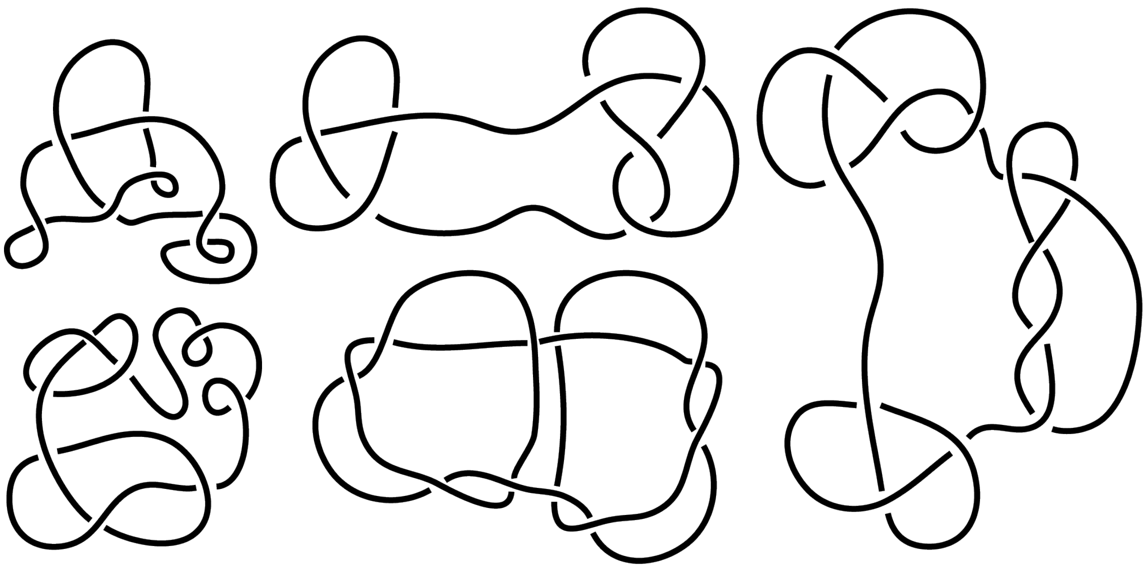


Composite Knots

A *composite* a knot that can be broken into prime knot components. For example, the following knot is two trefoils combined:

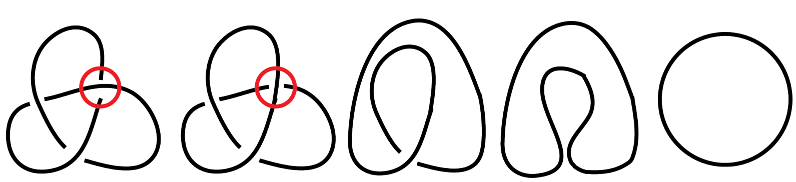


Exercise: Do your best to determine the prime knots that make up these composites.



Unknotting Number

We call these objects knots because they cannot be unravelled to reveal the unknot in any physical way. Everyday knots (like on your shoes) are designed for a similar purpose, they cannot be unravelled easily. What does it take to unknot a knot? The *unknotting number* of a knot is the fewest number of crossings that need to be switched from under to over or over to under to make the knot a projection of the unknot. It is another value, like the minimal crossing number, that is difficult to find but describes particular knots. For example, the trefoil knot has an unknotting value of one, because only one crossing needs to be reversed to make the projection equivalent to the unknot.



The unknotting number of a knot is difficult to find because, much like the minimal crossing number, we may not be certain that our projection can be unknotted in a smaller number of crossing changes. It is already difficult enough determining if a particular projection is the unknot, now we have to find the smallest number of crossing changes to guarantee the projection is an unknot. If two knots have different unknotting numbers, then they are certainly different knots. For simple knots, such as the primes we have seen, the unkotting number is not difficult to find.

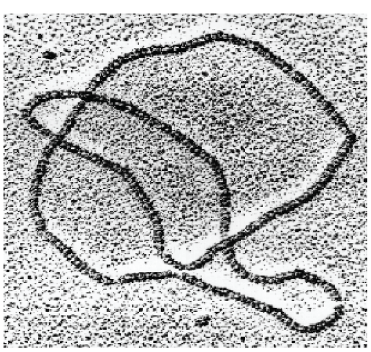
Exercise: For each prime and composite knot before this, determine the unknotting number.

Knots in the Real World

You have now seen several different examples of knots and two different invariants of a knot. The invariants – the minimal crossing number and unkotting number – are used to determine if two knots are different from each other. However, they do not work for any two knots we want to compare. Sometimes, knots can have the same invariants but then how do we tell these knots apart? There are other methods but none of them always works. In fact, this is still an unsolved problem in knot theory. No one has discovered an algorithm that takes in any two knots and determines if they are different or the same. Similarly, no one has discovered a method of finding out if a projection is the unknot.

Nonetheless, even difficult problems can have simple solutions. Maybe you will be the one to discover the secrets knots are hiding. Knots are plentiful in nature and in problems humans are trying to solve. Although mathematical knots differ from knots on shoelaces or for other tying purposes, their principles can still be applied to tying everyday knots.

Knots can also be found in every cell in your body. Research into the shape of DNA and how two metres of the stuff can fit inside every one of your body cells has found that it bunches itself into tight formations. Sometimes these formations are knots. The following image is a strand of DNA that forms a trefoil knot.



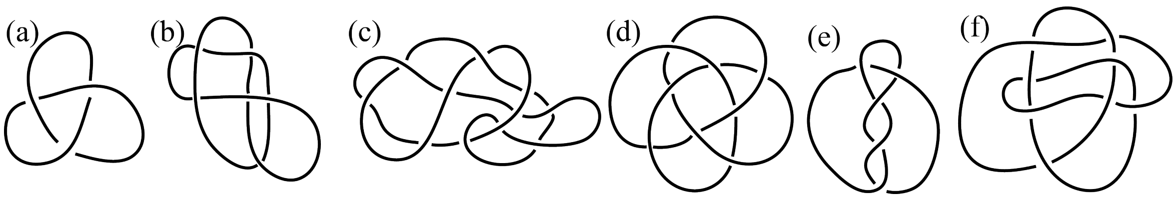
Your cells also make copies of each other, like when healing a cut. In order to copy DNA, the cell needs to unknot it so that it can easily separate the DNA in half – half for the new cell, half for the old. To do so, the process cuts the DNA strand and reattaches it after moving the strand around. Essentially, the DNA has an unkotting number and the replication process needs to find it or a number close to it.

Image Retrieved from: `www.popmath.org.uk/rpamaths/rpampages/knotsvirus.html`

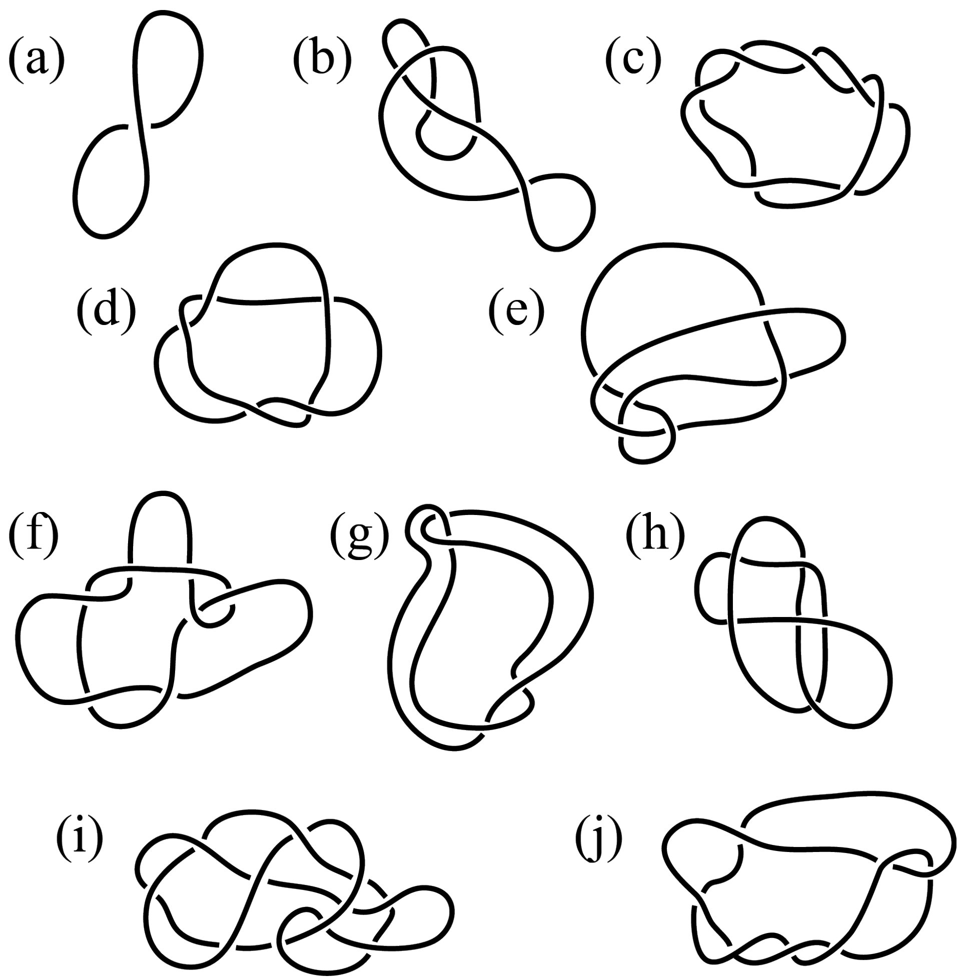
Exercises

For many of these exercises we will need a knotting tool such as a piece of string, pipecleaners, or surgical tubing.

1. An *alternating* knot is one in which the crossings are always switching from over to under and under to over. Determine which of the following knot projections are alternating:



2. The following projections are all prime knots that we have seen. Use your knotting tool of choice and determine what prime knot they really are (manipulate your knot to look like one of the projections of a prime knot we have seen).



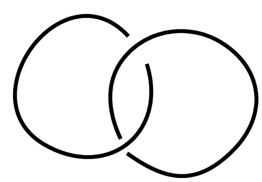
3. As was mentioned above, finding the minimal crossing number is really tough. Sophie has a particular knot she is untangling. She wants to determine what knot it is and she thinks the minimal crossing number will help.

   (a) If Sophie has found a projection with only four crossings, does she now know with certainty what knot it is?

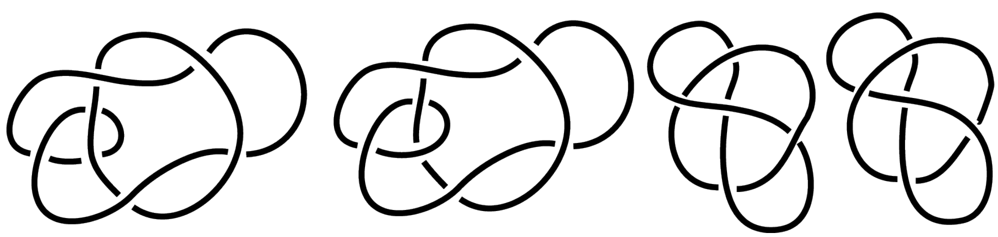
   (b) What if Sophie can manage to find a projection with only three crossings. Will she know with certainty what knot it is then as well?

   (c) How does this example illustrate why the minimum crossing number is tough to calculate?

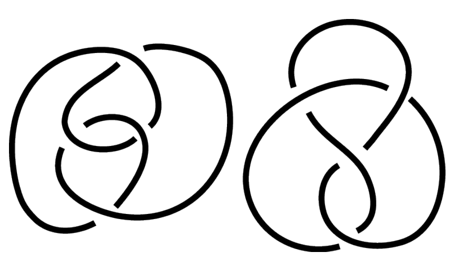
4. A *link* is two or more closed, connected loops. For example, here is a link:



   Which of the following are links and which are just unknots?



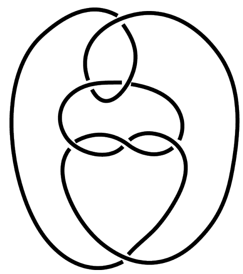
5. There are two popular projections of the figure-8 knot but they are both the same.



   (a) Using your knotting tools, find a way to transform one projection into the other.

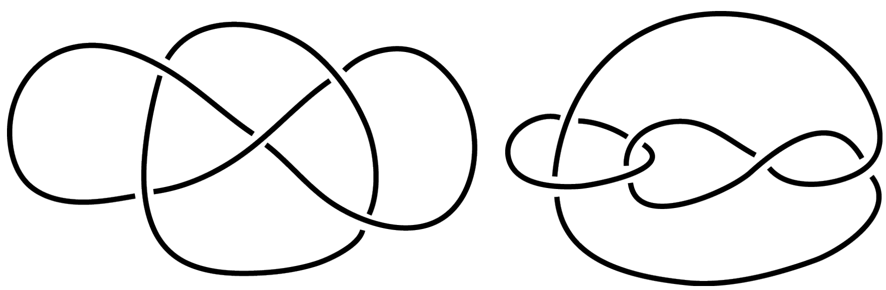
   (b) Pay close attention to what moves you make. Using Reidemeister moves, come up with instructions to turn one projection into the other.

6. Is this a knot or an unknot?



If it is the unknot, find a sequence of Reidemeister moves that will untangle it.

7. Here are two common projections of a link called the Whitehead link. Use your knotting tools to transform one projection into the other.



8. Show that all possible projections of knots with one crossing are the unknot.

9. * A trefoil knot is the simplest knot that is not *trivial* (or, in other words, is not the unknot). How do we know that the simplest nontrivial knot we can find must have three crossings and not, say, two?

10. 

(Take this problem home!) Crochet is a hobby that can produce many items, from scarves to mathematical figures. Watch this timelapse video of the creation of a mathematical surface out of yarn `https://youtu.be/EZ2Fw-mS8c0`. If you are interested in learning crochet, you should find some tutorials on YouTube. If you decide to try it then practice by making a simple chain bracelet. Is this simple chain a knot or an unknot (if we attached the two ends together)? To the left are examples of what crochet can produce.