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Grade 7/8 Math Circles
Greek Constructions - Solutions

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Mathematics Without Numbers

The influences of Ancient Greece are still being felt in our lives today. Democracy, deductive reasoning, and the Olympics are all elements of our lives that have their roots in Ancient Greece. Similarly, much of our mathematics has its roots in the work of Greek mathematicians such as Euclid and Archimedes. For example, Euclid established our basic understanding of geometry in his work the *Elements* and Archimedes generated one of the most accurate estimates of Pi ever. What makes these accomplishments so extraordinary is that the Greeks had no number system. The Greeks interpreted all of their numbers as lengths or areas of geometric shapes.

For example,

$$x(a + b) = xa + xb$$

would become

If there are two straight lines, and one of them is cut into any number of segments, then the rectangle contained by the two straight lines equals the sum of the rectangles contained by the uncut straight line and each of the segments.

While this is clearly confusing to us today, this is how the Greeks did the vast majority of their mathematics - work that we still rely on to this day.

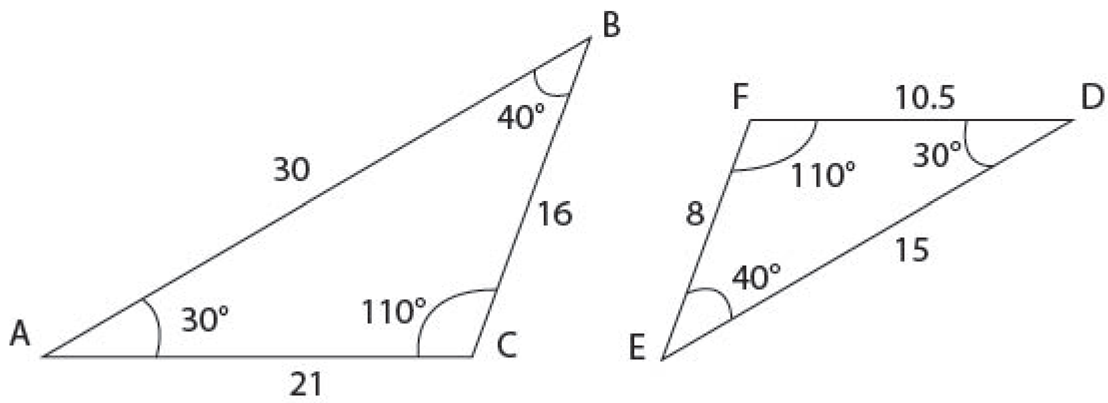
Basic Properties of Circles & Triangles

The majority of the Greek constructions we will be exploring today are based on triangles and circles. The following section is a brief overview of the properties that will be important for this week's lesson.

Similar Triangles

We say that two triangles are similar if the corresponding angles are equal and the ratios of corresponding side lengths are equal. What this means is that one triangle is essentially a copy of the other that has been made larger or smaller. If $\triangle ABC$ and $\triangle DEF$ are similar, we write $\triangle ABC \sim \triangle DEF$

To see this more clearly, consider $\triangle ABC$ and $\triangle DEF$ below.



These triangles are similar triangles. We can see this as

$$\angle ABC = \angle DEF$$
$$\angle BAC = \angle EDF$$
$$\angle ACB = \angle DFE$$

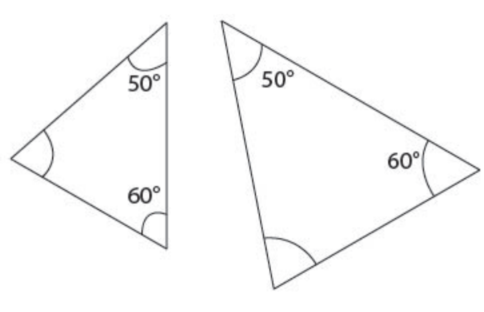
and

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = 2$$

If we have all of the angles and side lengths in a pair of triangles, it is easy to check whether or not they are similar. But what if we don't have all of this information? How can we determine whether or not two triangles are similar?

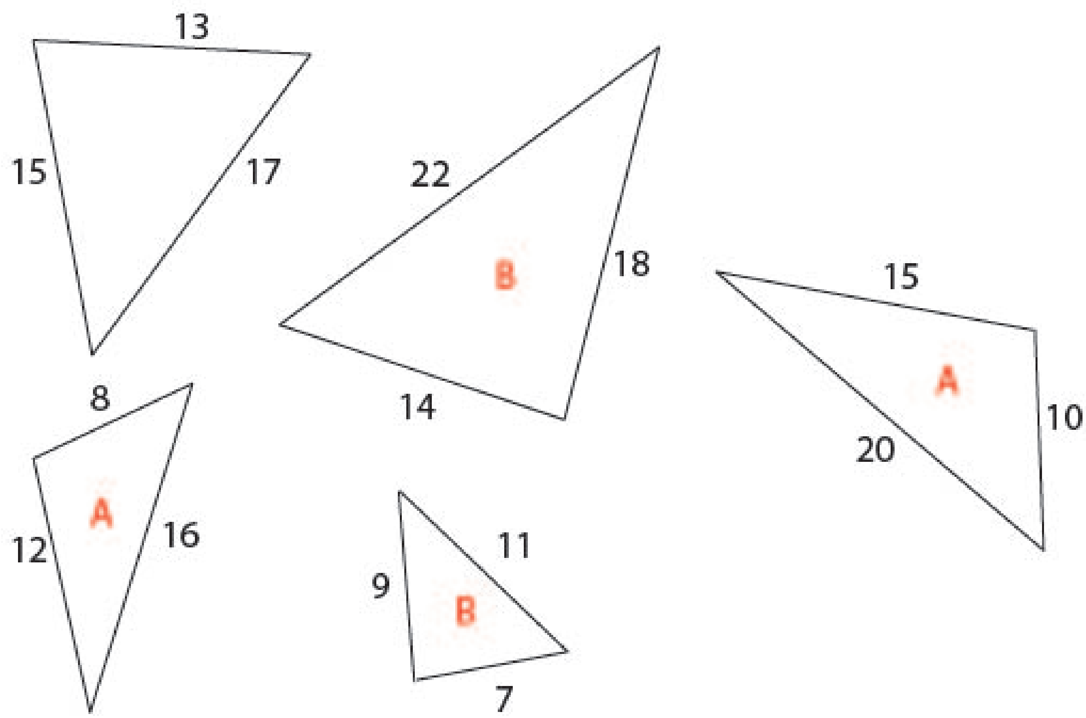
Method 1: Angle-Angle-Angle

The first way to determine if two triangles are similar is when we know the angles in each triangle. If these angles are the same in both triangles, then the two triangles are similar. Note that we don't need to know all three angles right away. Since the sum of the angles in a triangle is always 180°, we can find the third angle with a little bit of simple subtraction. This is why we can quickly see that the triangle below are similar.



Method 2: Side-Side-Side

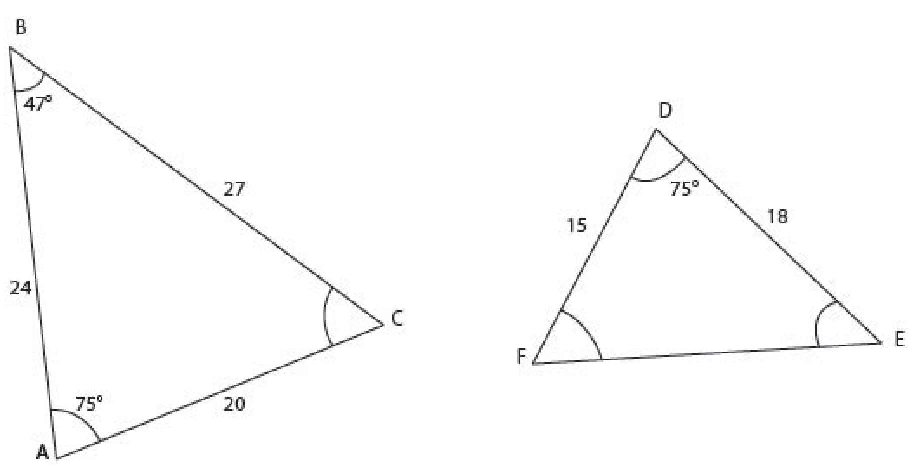
We can also determine if two triangles are similar if we know all of the side lengths. Two triangles are similar if the ratio between corresponding side lengths are the same. Try it out for yourself on the five triangles below. Circle each group of similar triangles. If there is more than one group, label each group with a different letter.



Method 3: Side-Angle-Side

The last method we will talk about for determining triangle similarity is the Side-Angle-Side method. If two triangles have two sides with proportional lengths, and the angle between is the same, then they are similar.

To put all of these ideas together, let's look at the following pair of triangles.



We can see that both triangles have an angle of 75°, so now we can look at the ratio of side lengths between the two triangles. From this we see that:

$$\frac{DF}{AC} = \frac{15}{20} = \frac{3}{4}$$

and

$$\frac{DE}{AB} = \frac{18}{24} = \frac{3}{4}$$

Since the ratios are the same, and the angle is the same, then by our Side-Angle-Side rule we know these triangles are similar.

But what can we do with this knowledge?

Knowing that two triangles are similar allows us to calculate the other unknown angles and sides in each triangle.

In our example, we don't know how long side EF is or what $\angle DEF$ is in $\triangle DEF$

Since our two triangles are similar we know that the ratios between corresponding side lengths are the same. In other words:

$$\frac{DE}{AB} = \frac{18}{24} = \frac{3}{4}$$

and

$$\frac{DF}{AC} = \frac{15}{20} = \frac{3}{4}$$

and since $BC = 27$, we know

$$EF = 27 \times \frac{3}{4} = 20.25$$

We also know that the angles in similar triangles are equal. This means that

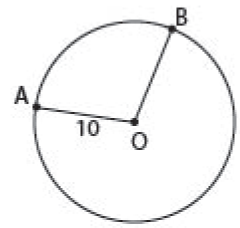
$$\angle DEF = \angle ABC = 47^\circ$$

Circles

This section is a brief review of the most important property of circles for today - the radius.

Definition: A *radius* is a straight line from the centre of a circle to a point on its circumference (the boundary line of the circle).

Recall that all circles can be described by a centre point and the length of their radius. A circle is simply all points (infinitely many!) that are exactly the distance of the radius from the centre of the circle. For example, in the circle below points A and B are both the same distance from the centre point O.



Euclidean Rules of the Game

When the Greeks did their math they relied on what we now call *constructible numbers*. A number x is constructible if we can make a line segment of length x in a finite number of steps using only a compass, a straightedge, and the following rules:

- You may draw a circle around any centre, and with any radius.
- You can draw a straight line between any two points
- You can find the intersection points of two lines, two circles, or one line and one circle

Constructing Triangles & Line Segments

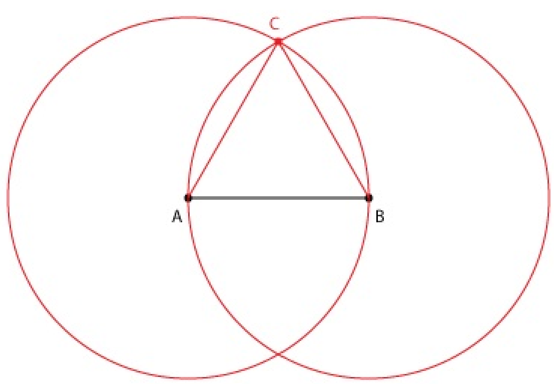
The first construction we will cover are from the beginning of Euclid's *Elements*. These are some of the building blocks we can use for more complex constructions later on.

Equilateral Triangles *Proposition I.1:* Given a line AB, construct an equilateral triangle $\triangle ABC$ whose side lengths are the length of AB.

Step 1. Construct a circle centred at A, with radius AB

Step 2. Construct a circle centred at B with radius AB

Step 3. Label the point where the circles intersect as C. $\triangle ABC$ is an equilateral triangle.



Moving Straight Lines *Proposition I.2:* Given a line BC and a third point A, construct a line segment of length BC which has A as an endpoint.

Step 1. Draw a straight line between points A and B

Step 2. Using the equilateral triangle construction, construct the equilateral triangle $\triangle DAB$

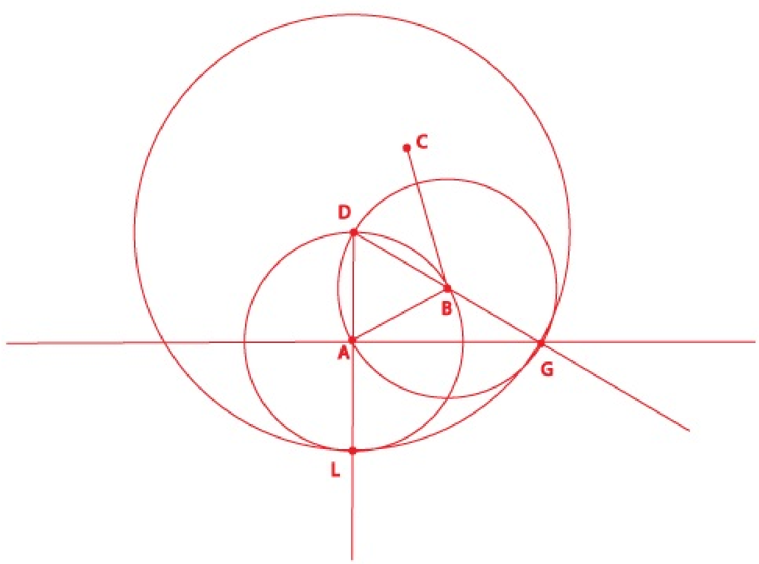
Step 3. Extend the lines DA and DB - Make them fairly long, but the precise length isn't important

Step 4. Draw a circle with centre B and a radius of length BC

Step 5. Label the intersection of this circle and the line DB as point G

Step 6. Draw a circle with centre D and radius DG

Step 7. Label point L as the intersection of that circle and line DA. Then, line AL is the same length as line BC, and one endpoint is A



Take a minute to think about that last exercise. Can you see why this construction works? It took Euclid that much work just to move a line. Today, we could simply measure the line we want to move, and then move it - easy. Now that you can see how much work went into even simple mathematics in Ancient Greece, we can take a few short cuts. For the rest of the problems, to move a line, simply use your compass to measure and move it to a new endpoint.

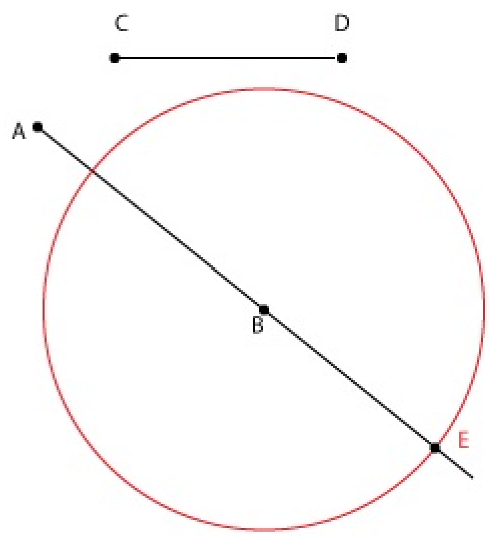
Addition & Subtraction: With the basic tools from the first two constructions, addition and subtraction become straight-forward tasks.

Addition: Given two line segments AB and CD construct a line segment that is $AB + CD$ units long.

Step 1. Draw a circle centred at B with a radius of CD

Step 2. Extend the line AB until it intersects with the circle. Label the intersection point E (Note: Point E should not be between A and B)

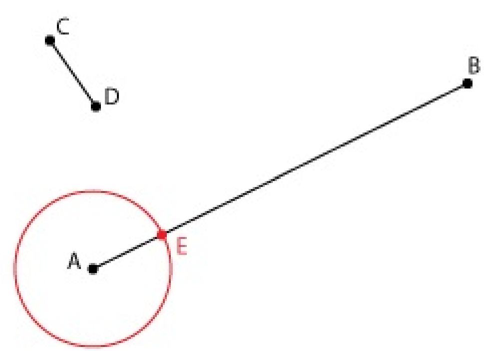
Step 3. The line segment AE is the desired line segment



Subtraction: Similar to addition, subtraction is a simple thing once we know how to move lines around. One thing to keep in mind though is that the Greeks didn't have negative numbers or zero,and so they always subtracted the smaller number from the larger one. So if a Greek had 5 and 8 would always calculate $8 - 5 = 3$ and never $5 - 8 = -3$.

For this example, line AB will be the longer line, and we are subtracting the line CD

1. Draw a circle centred at A with radius CD
2. Label the intersection of this circle and AB as point E
3. The line segment EB has length $AB - CD$



Multiplication & Division: Using what we know about similar triangles, we can geometrically multiply and divide. First, we need to create parallel line. After that, all we need are line segments AB and CD whose length we want to multiply or divide, and a line segment of unit length.

Parallel Lines: Suppose we have a line AB and a point C. Our goal is to draw a line that is parallel to AB that goes through point C. Note: When you have a choice of intersection points, pick the closest point to C that is on the same side (above or below) of AB.

Step 1. Draw a circle centred at A with radius AC

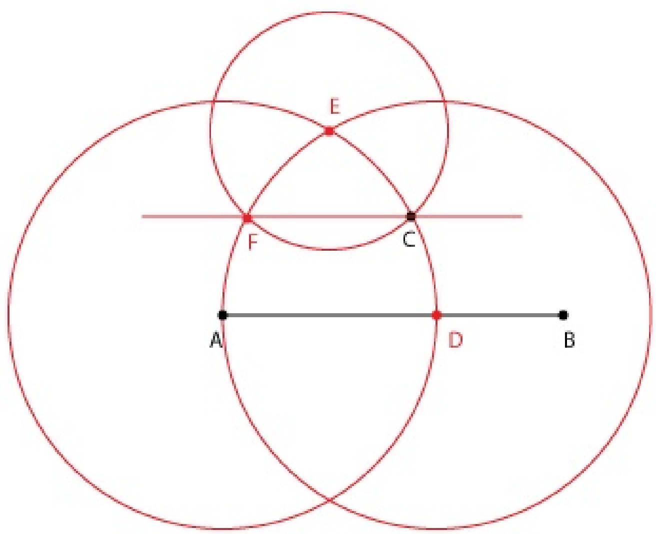
Step 2. Label the intersection point of this circle and AB as point D. You may need to extend AB to find the point D

Step 3. Draw a circle centred at D with radius AD

Step 4. Label the intersection point of these circles (the one closest to C) as point E

Step 5. Draw a circle centred at E with radius CE

Step 6. Label the intersection of this circle and the circle centred at D as point F. The line CF is parallel to AB and goes through point C



Now that we know how to construct parallel lines, we will just draw them when needed rather than repeating the entire construction. To draw a parallel line in this way, place your straight edge along the line AB, and then slide it along your paper until you reach point C.

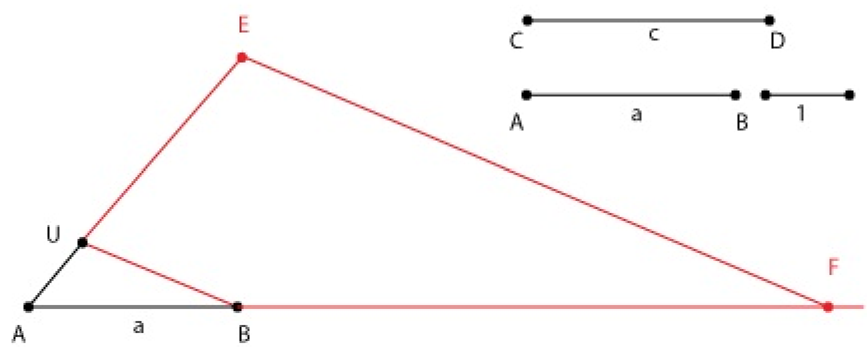
Multiplication: Our goal is to construct a line segment of length ac given a line segment AB of length a, a line segment CD of length c, and a line segment U of unit length (length 1).

Step 1. Draw a triangle where one side is AB and one side is U. We will call this triangle $\triangle AUB$. This step has been started in the diagram below. You only need to finish the triangle by drawing the line segment UB

Step 2. Using the addition construction, add CD to AU. Call the end point of the new line E

Step 3. Draw a line parallel to UB through the point E. Extend the line AB until these two lines meet. Call the intersection point F

Step 4. The length of the line segment BF is ac



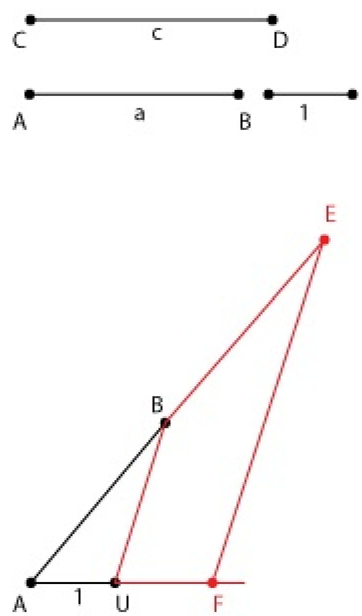
Division: Our goal is to construct a line segment of length $\frac{c}{a}$ given a line segment AB of length a, a line segment of length c, and a line segment U of unit length (length 1).

Step 1. Draw a triangle where one side is AB and one side is U. We will call this triangle $\triangle AUB$. This step has been started in the diagram below. You only need to finish the triangle by drawing the line segment UB

Step 2. Using the addition construction add CD to AB. Call the end point of the new line E

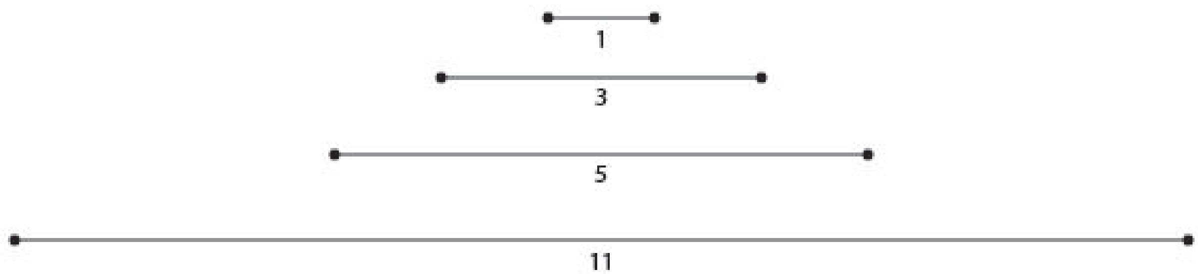
Step 3. Draw a line parallel to the line BU through the point E. Extend the line AU until these two lines meet. Call the intersection point of these lines F

Step 4. The length of the line segment UF is $\frac{c}{a}$



Problem Set

The following problems ask you to construct lines, triangles, and circles of given lengths/radii. You can use any tool covered above to help you, and you can use any line or circle that you construct below. For example, if you construct a line segment of length 2, you can use that line segment in later problems. There are many different ways to accomplish these constructions - try to use the fewest steps! To get you started, lines of various lengths are included below.

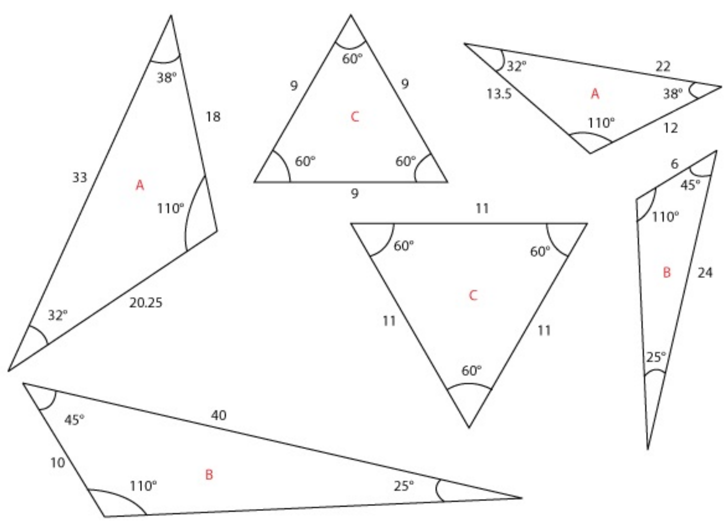


1. *Similar Triangles:* Use the methods for deciding whether triangles are similar or not to help with this problem. All parts of this question relate to the triangles depicted below.

   (a) Match up the pairs of similar triangles. Label each pair with the method you used to determine that the triangles are similar.

   (b) Using the similarity of the triangles, fill in the missing side lengths and angle measurements for each pair of triangles

   (c) What can you say about similarity and equilateral triangles? Are equilateral triangles always similar? Why or why not? Equilateral traingles are all similar to each other. Since each equilateral traingle has the same angles (all 60°) and all the side lengths are the same in each triangle, so the ratio between side lengths will be the same.



Solutions to the constructions will vary depending on the student. Sample arithmetic for each question is provided.

2. *Addition & Subtraction:* For each problem, try constructing the solution once with addition, and once with subtraction. Remember, you can use the lines you create in the first problems to help with the later ones.

   (a) Construct a line segment of length 7 e.g. $5 + 1 + 1$

   (b) Construct a line segment of length 2 e.g. $3 - 1$

   (c) Construct a line segment of length 20 e.g. $11 + 7 + 2$ using the lines from parts a and b

3. Construct an equilateral triangle with side length 5

4. Construct a line segment with length 3×5

5. Construct a line segment with length $\frac{11}{3}$

6. Draw a line of length 5 and a point not on that line. Using the full construction, draw a line of length 5 through your point.

7. Create an equilateral triangle with side lengths $\frac{7}{2}$

8. Construct a circle with radius $\frac{9}{4}$

9. Construct a triangle where one side length is $\frac{26}{9}$ and another side is length $\frac{31}{11}$

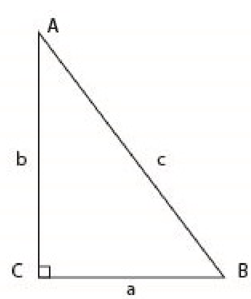
10. Draw a random line, and a point not on that line. Construct a new line that is parallel to your line, is exactly half the length of you line, and passes through your point

Challenge Problems

The following problems are this week's challenge problems. They involve a few new constructions, and you will need the Pythagorean Theorem which is stated here without proof.

Additional Constructions Pythagorean Theorem: In a right triangle $\triangle ABC$ with side lengths $a, b, \& c$, with c being the hypotenuse (see diagram below), then the side lengths fit the following equation:

$$c^2 = a^2 + b^2$$



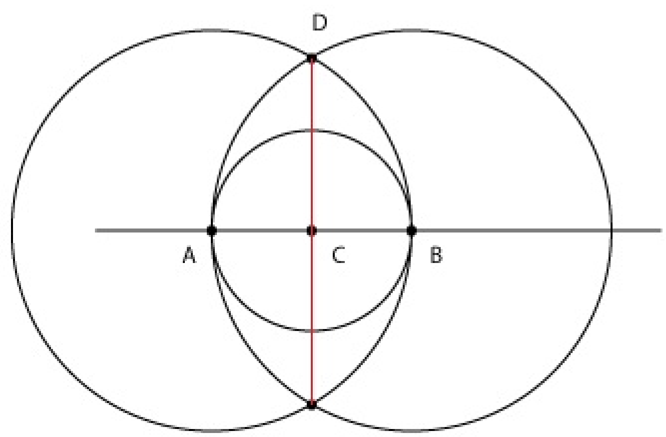
Perpendicular Lines: Our goal here is to construct a line that is perpendicular to a given line that passes through a point C on that line

Step 1. Draw a circle centred at C with a radius of 1

Step 2. Label the points where the circle intersects the given line A and B

Step 3. Draw two circles, both with radius AB with one centred at A and the other centred at B

Step 4. Label the intersection of these circles point D. Then, the line segment CD will be perpendicular to the original line, and pass through C.



Square Roots: Given a line AB of length x, we want to construct a line segment of length $\sqrt{x}$

Before looking at the steps listed below, try to come up with a method on your own. If you think you have found your own method, think about how you could prove to someone else that your method always works.

Step 1. Extend line AB to a point C so that $BC = 1$

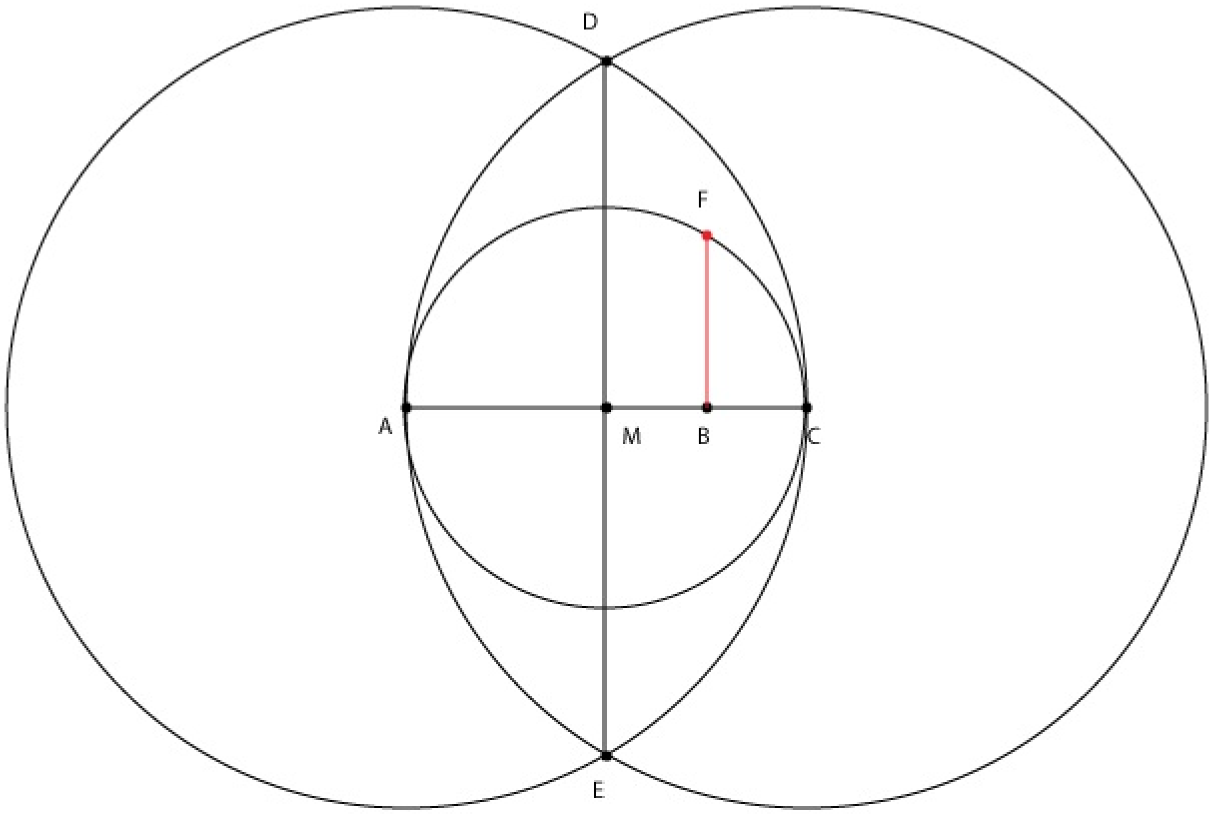
Step 2. Draw two circles, one centred at A and the other centred at C, both with radius AC. Label the intersection points of these circles as the points D and E

Step 3. Construct the line DE and mark the intersection of DE and AC as point M

Step 4. Draw a circle centred at M with a radius of length AM

Step 5. Construct a line through point B that is perpendicular to AC

Step 6. Label the intersection of this line and the circle centred at M as point F. The line BF has length $\sqrt{x}$



Additional Problems Solutions to the constructions will vary from student to student

1. Using your knowledge of parallel and perpendicular lines, construct a square with side lengths 2
2. Construct an equilateral triangle with side lengths $\sqrt{2}$
3. Show that the length of BF in our multiplication construction really is $a \times c$ (Hint: Similar triangles!)

   $\triangle AUB$ is similar to $\triangle AEF$. This means the ratios of the side lengths are the same. In other words:

$$\frac{AE}{AU} = \frac{AF}{AB}$$
$$AB = a$$
$$AU = 1$$
$$AE = c + 1$$
$$AF = a + BF$$

Combining these facts we know that:

$$\frac{c+1}{1} = \frac{a+BF}{a}$$

cross multiplying gives us:

$$ac + a = a + BF$$

Simplifying gives $BF = ac$ as required

4. Show that the length of UF from our division construction is $\dfrac{c}{a}$

$\triangle AUB$ is similar to $\triangle AEF$. This means the ratios of the side lengths are the same. In other words:

$$\frac{AF}{AU} = \frac{AE}{AB}$$
$$AB = a$$
$$AU = 1$$
$$AE = a + c$$
$$AF = UF + 1$$

Combining these facts we know that:

$$\frac{UF+1}{1} = \frac{a+c}{a}$$

cross multiplying gives us:

$$a(UF) + a = a + c$$

Simplifying gives

$$a(UF) = c$$

This yields $UF = \frac{a}{c}$ as required

5. Show that the line BF in the above construction is of length $\sqrt{x}$

From our construction, we know the following lengths:

$$AB = x$$
$$BC = 1$$

M is the midpoint of AC. This means that:

$$AM = \frac{x+1}{2}$$

and

$$MB = (x - \frac{x+1}{2}) = \frac{x-1}{2}$$

We notice that both A and F lie on the same circle centred at M. This means

$$AM = FM = \frac{x+1}{2}$$

$\triangle MBF$ is a right-triangle, and so by the Pythagorean Theorem we know

$$FM^2 = BF^2 + MB^2$$

If we substitute the side lengths we know into this equation and simplify, we have:

$$\begin{aligned}
(\frac{x+1}{2})^2 &= BF^2 + (\frac{x-1}{2})^2 \\
BF^2 &= \frac{x^2+2x+1}{4} - \frac{x^2-2x+1}{4} \\
BF^2 &= x \\
BF &= \sqrt{x}
\end{aligned}$$