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Centre for Education in
Mathematics and Computing

Grade 11/12 Math Circles
Conics & Applications
The Mathematics of Orbits

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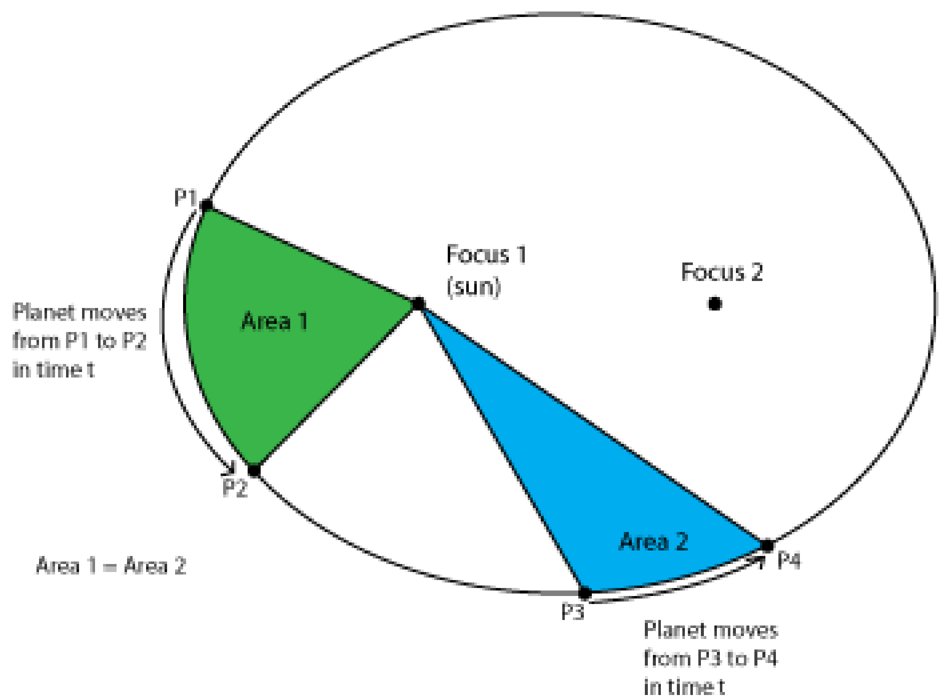
What is applied mathematics?

Applied Mathematics is the application of mathematics which can occur in any area of science, economics, engineering, etc.

Historically, Applied Mathematics grew out of a desire to understand the physical world. For example, Kepler's Laws examined how the planets move around the sun.



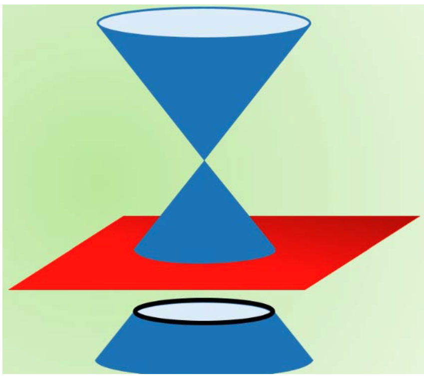
Einstein invented the general theory of relativity by applying methods from differential geometry. Other applications of Applied Mathematics included modeling the formation of the brain, tumor growth, carbon nanotubes, and fluid mechanics.

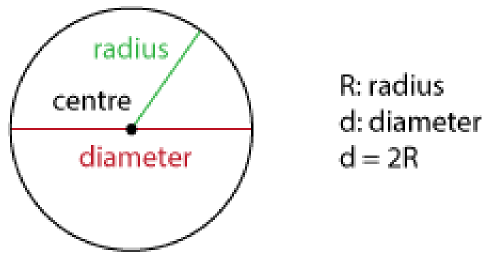
Conic Sections

A conic section is a curve formed by the intersection of a plane and a double cone. By changing the angle and location of this intersection, we can produce different conics. What conic sections are there?

Circles

A circle can be formed by the intersection of a cone and a plane that is perpendicular to the axis of the cone.





The diameter is the length of a line segment passing through the center whose endpoints are on the circle.

A circle is the set of points (in a plane) equidistant from a given point. This point is called the *centre*. The circle is all points that are R distance away from the centre.

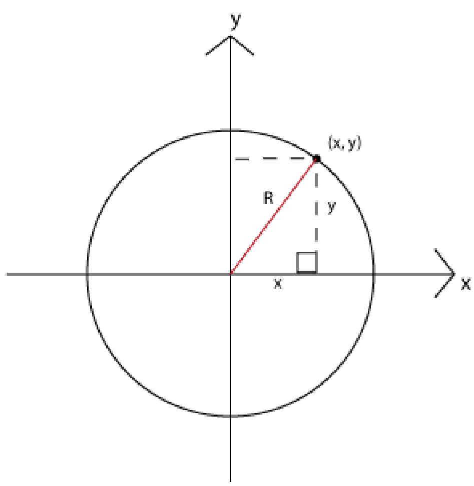
Equation of a circle

Suppose we put the centre at point $(0, 0)$. So the circle is all the points (x, y) that are R units away from the centre $(0, 0)$.

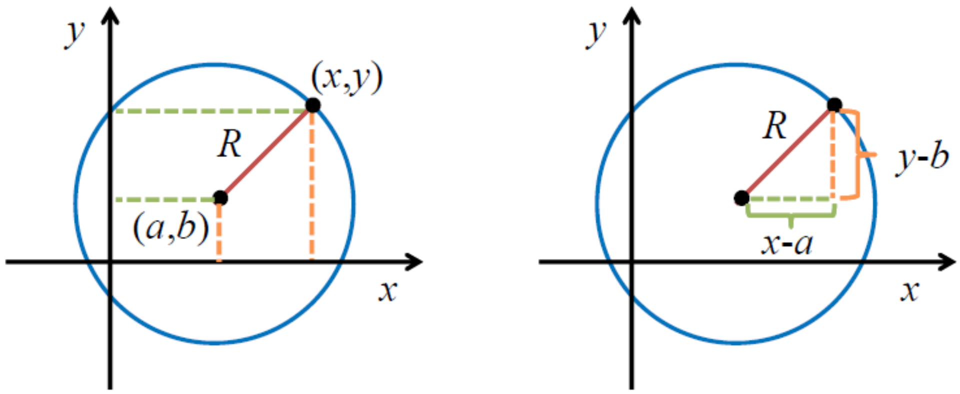
We make a right-angled triangle as shown in the diagram below, and apply the Pythagorean Theorem. We can then see that:

$$x^2 + y^2 = R^2$$

If $R = 1$, then the circle is called a *unit circle*.



Suppose now that we put the centre of the circle at the point (a, b). Then, the circle would be all the points (x, y) that are R units away from (a, b).



Again we can apply the Pythagorean Theorem to obtain the following result:

$$(x - a)^2 + (y - b)^2 = R^2$$

This is the *standard form* for the equation of a circle. We can also write it in the following *general form*:

$$x^2 + y^2 + Ax + By + C = 0$$

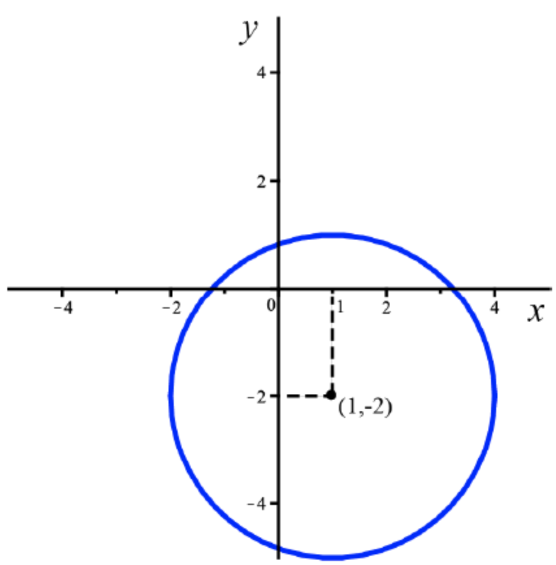
Example:

$a = 1, b = -1, R = 3 : \quad (x - 1)^2 + (y + 2) = 3^2$

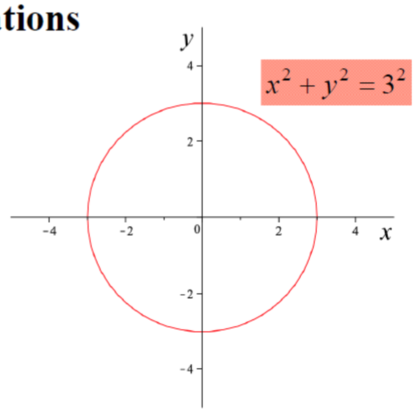
Expand: $\quad x^2 - 2x + 1 + y^2 + 4y + 4 = 9$

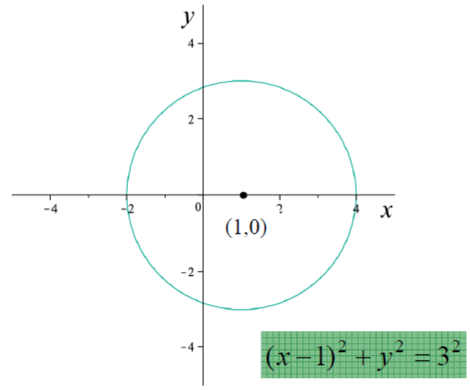
Rewrite: $\quad x^2 + y^2 - 2x + 4y + 1 + 4 - 9 = 0$

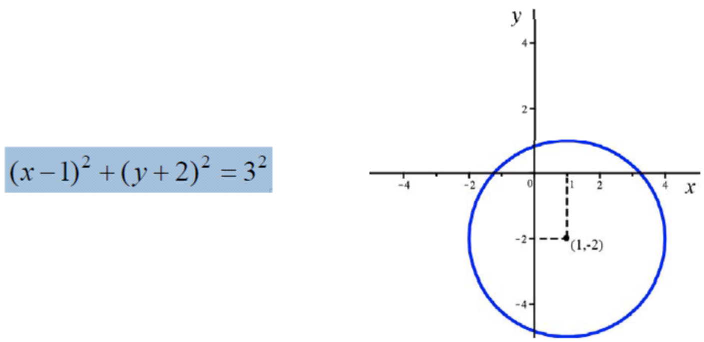
Simplfiy to obtain: $\quad x^2 + y^2 - 2x + 4y - 4 = 0$



Translations





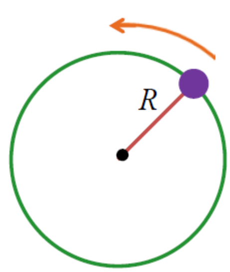


Moving from General Form to Standard Form

General Form	$x^2 + y^2 - 2x + 4y - 4 = 0$
Collect x and y terms	$(x^2 - 2x) + (y^2 + 4y) = 4$
Complete the square for x	$(x^2 - 2x + (-1)^2) + (y^2 + 4y) = 4 + (-1)^2$
Complete the square for y	$(x^2 - 2x + (-1)^2) + (y^2 + 4y + (2)^2) = 4 + (-1)^2 + (2)^2$
Simplify	$(x^2 - 2x + 1) + (y^2 + 4y + 4) = 4 + 1 + 4$
Standard Form	$(x-1)^2 + (y+2)^2 = 3^2$

The Math Behind Circular Motion

Circulate motion has many applications, including a ball being spun around by a string, and the rotation of a spinning wheel. The motion of the moon around the earth is *nearly* circular.



Uniform Circular Motion

- Period (T): Time to go around once (s)
- Frequency (f): Number of rotations per second ($\frac{1}{s}$)
- Angular velocity(ω): How many radians per second ($\frac{rad}{s}$)

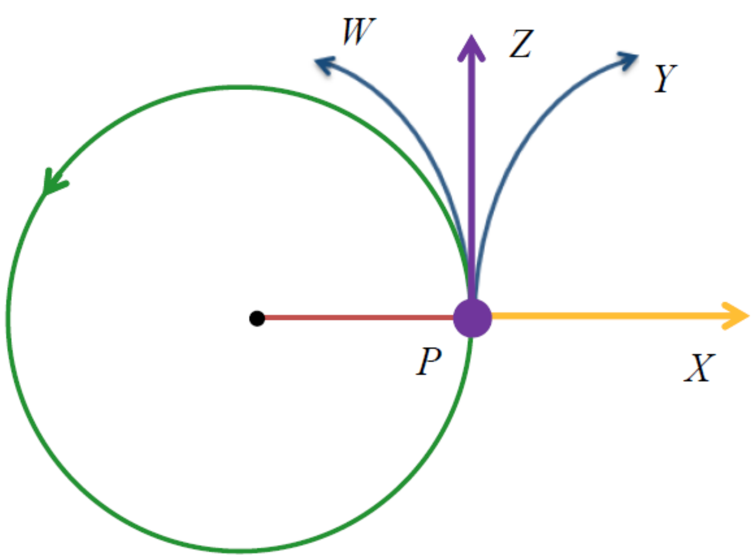
There are 2π radians in one full circle, and it takes T seconds to go around once. This means:

$$\omega = \frac{2\pi}{T}$$

The speed is given by

$$v = \frac{2\pi R}{T} = R\omega$$

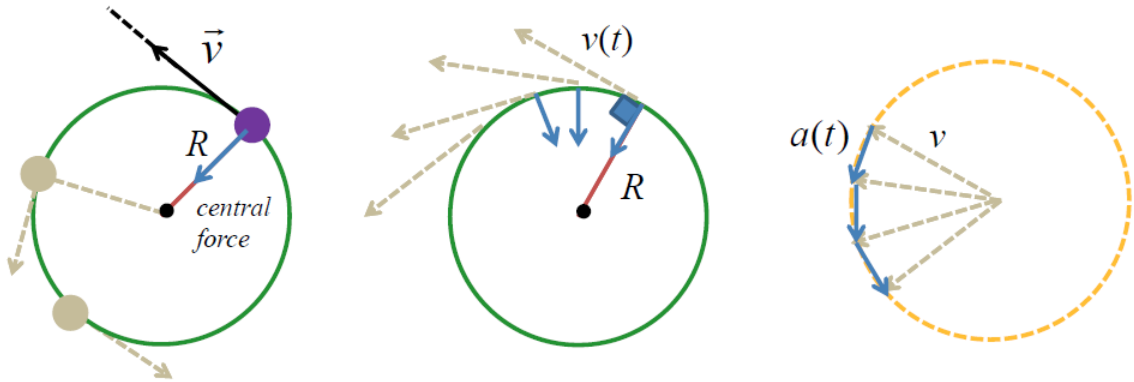
Question:
A student is spinning a ball on the end of a rope in a circular path. If the rope snaps at point P, which path does the ball follow?



What happens if we cut the rope?
In uniform circular motion, the speed is constant. However, the velocity is changing.

The direction of velocity is always tangent to the circle - there is no displacement in the radial direction.

The velocity is changing, thus there must be some acceleration. There is a non-zero radial component of the acceleration (central force).



$$\frac{2\pi R}{v} = \frac{2\pi v}{a} \rightarrow a = \frac{v^2}{R} = R\omega^2$$

Orbit of the Moon

The motion of the Moon around the Earth is *approximately* circular. We can estimate the time T the Moon takes to complete one circle around the Earth.

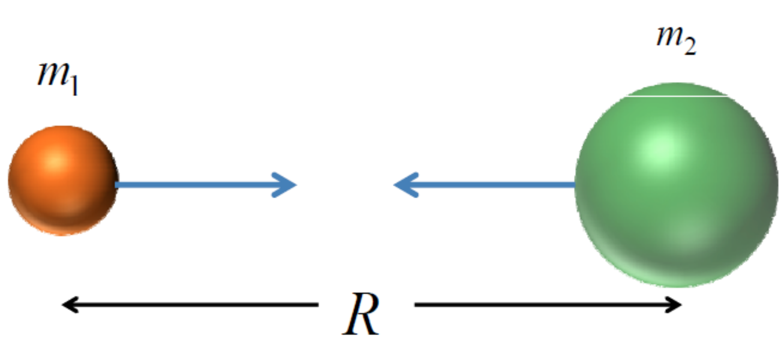
If an apple is drawn to the ground by the gravitational attraction of the Earth, why does the Moon - which is more or less spherical like an apple, (and definitely heavier) - not fall to the Earth?

Sir Isaac Newtown once said "I deduced that the forces which keep the planets in their orbs must [be] reciprocally as the squares of their distances from the centres about which they revolve." In other words, $F \sim \frac{1}{R^2}$

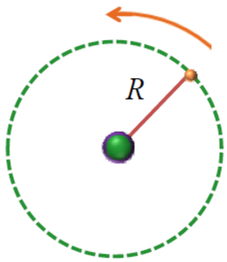
Gravitational Force

The force of gravitational attraction between *two* masses:

$$F = G\frac{m_1 m_2}{R^2}$$



Assuming the motion of the Moon around the Earth is circular:



$$F = m_M \underbrace{R\omega^2}_{\text{acceleration}} = G\frac{m_M m_E}{R^2}$$

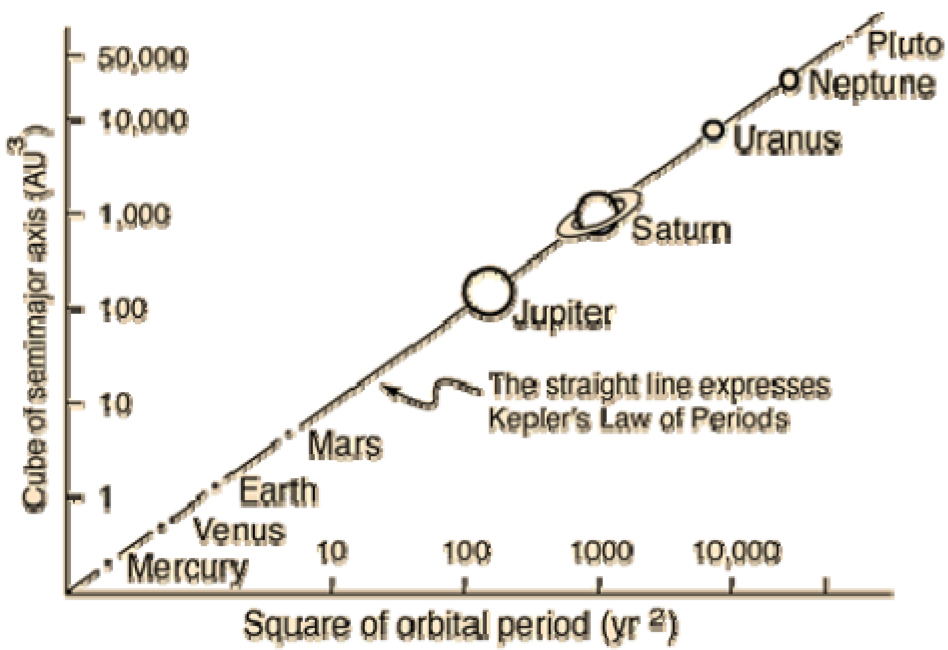
$$G\frac{m_M m_E}{R^2} = m_M R(\frac{2\pi}{T})^2 \rightarrow T = \sqrt{\frac{4\pi^2 R^3}{G m_E}}$$

Note that this period is independent of the mass of the Moon.

Kepler's Third Law

The square of the orbital period is proportional to the cube of the mean distance from the Sun. That is:

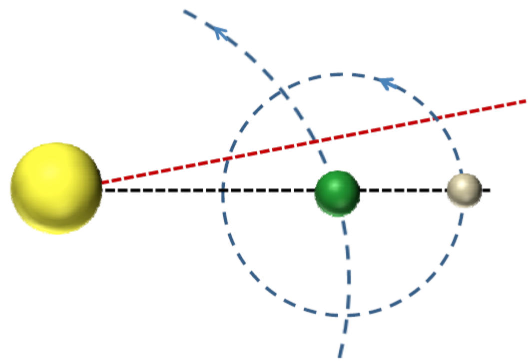
$$T^2 \sim R^3$$



Using $m_m = 7.36 \times 10^22$, $m_E = 5.98 \times 10^24 kg$, and $R = 3.82 \times 10^8 m$ for the distance between the Moon and the Earth, we obtain

$$T = \sqrt{\frac{4\pi^2 R^3}{G m_E}} = 2.35 \times 10^6 \text{ seconds} = 27.2 \text{ days}$$

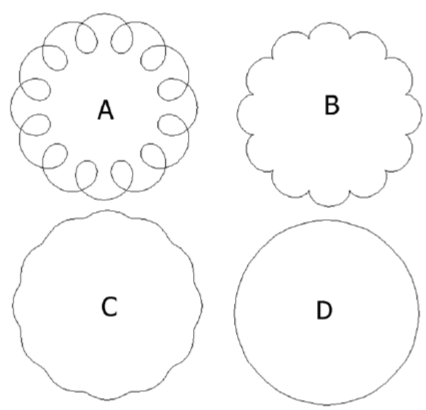
NB: The actual time is longer (with respect to the Sun) because the Earth is traveling around the Sun.



Thus, the time T_1 between consecutive full moons is approximately $T_1 \sim T = \Delta T$ where $\Delta T = \dfrac{T}{12} = 2.3$ days. So $T_1 \sim 29.5$ days.

The motion of the Moon around the Sun

We can assume that the orbits of the Earth around the Sun and the Moon around the Earth are both circles. Which of the following orbits *best* approximates the motion of the Moon around the Sun?



The radius of the Earth's orbit is about 400 times the radius of the Moon's orbit. The speed of the Earth around the Sun is about 30 times the speed of the Moon around the Earth.

Upper limit on speeds

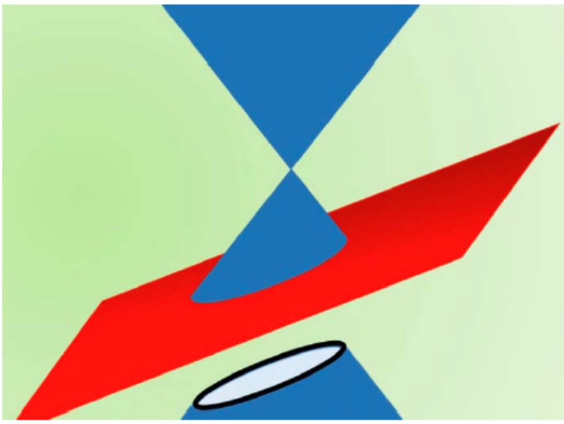
Sunlight takes about 8 minutes 17 seconds to travel the average distance from the surface of the Sun to the Earth.

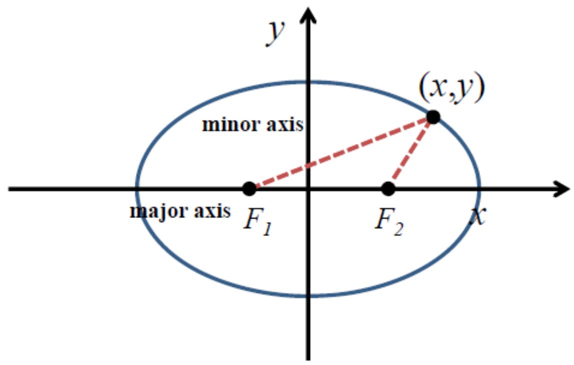
The geometric theory of gravitation

Objects warped the space-time around it causing it to become curved. As a result, objects experience gravitational attraction to each other.

Ellipses

An ellipse can be formed by the intersection of a cone and a tilted plane as follows.

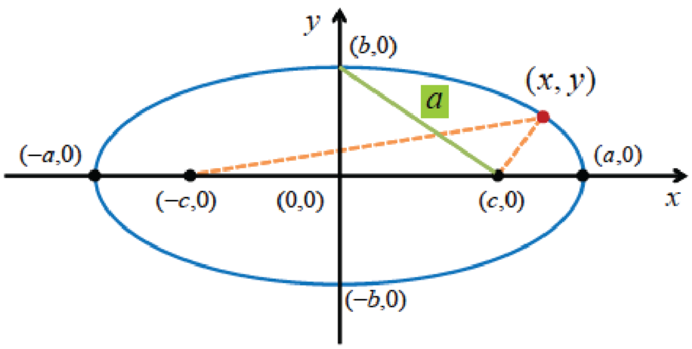




Ellipses have two axes (minor and major). The two foci of an ellipse are two special points F_1 and F_2

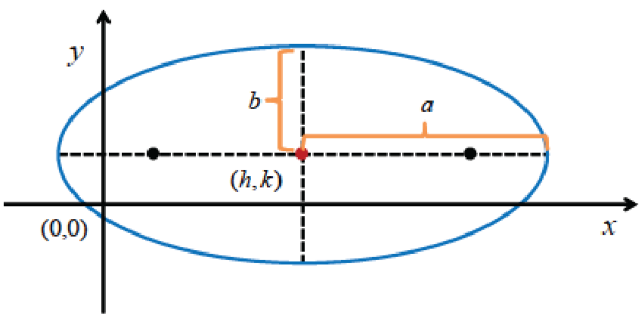
An ellipse is the set of points (in a plane) in which the sum of the distances from two fixed points - called foci - is equal to a constant.

Equation of an ellipse:



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$a^2 = c^2 + b^2$$



$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

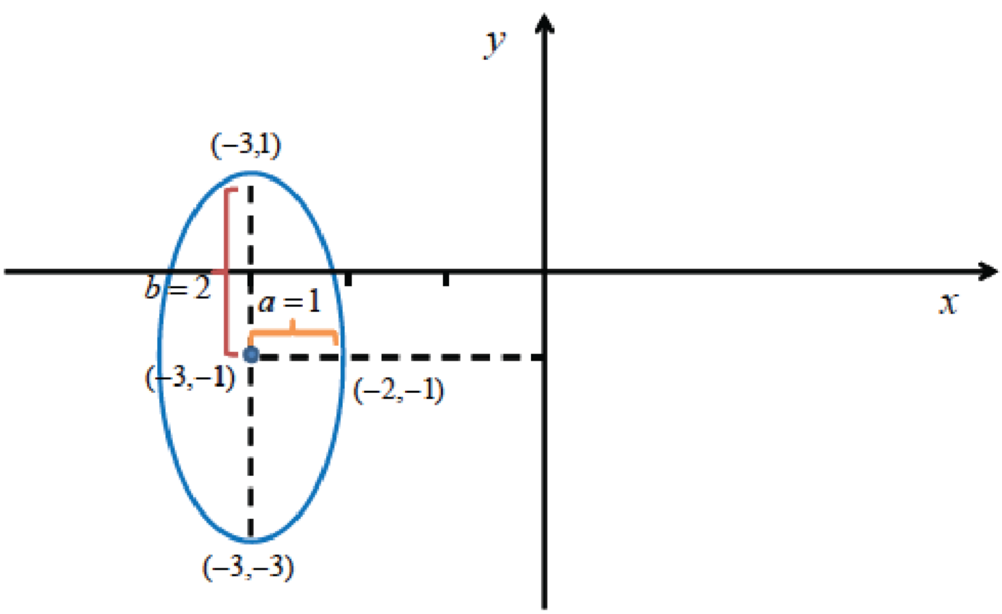
Example: Sketch $4x^2 + y^2 + 24x + 2y + 33 = 0$

General Form	$4x^2 + y^2 + 24x + 2y + 33 = 0$
Put x and y terms together	$4(x^2 + 6x) + (y^2 + 2y) = -33$
Complete square for x	$4(x^2 + 6x + 3^2) + (y^2 + 2y) = -33 + 4(3)^2$
Complete square for y	$4(x^2 + 6x + 3^2) + (y^2 + 2y + 1^2) = -33 + 4(3)^2 + 1^2$
Simplify	$4(x+3)^2 + (y+1)^2 = 4$
Standard form	$\frac{(x+3)^2}{1} + \frac{(y+1)^2}{4} = 1$

$a = 1$, $b = 2$, center $(-3, -2)$

A vertical ellipse.

Resulting Graph of $\frac{(x+3)^2}{1} + \frac{(y+1)^2}{4} = 1$:



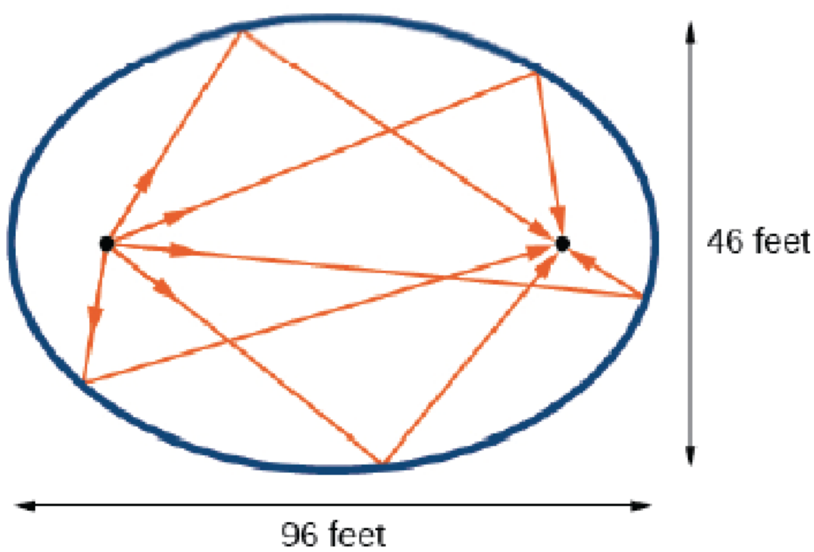
Acoustic properties of an ellipse: Whispering Chamber

If you stand at one end of the room, you can hear a whisper from a person standing at the other end.

The National Statuary Hall in Washington, D.C. (Greg Palmer, Flickr).

Example:

The Statuary Hall in the Capitol Building in Washington, D.C. is a whispering chamber. Its dimensions are 46 feet wide by 96. If two senators standing at the foci of this room can hear each other whisper, how far apart are the senators? Round to the nearest foot.



Solution:

We are assuming a horizontal ellipse with centre $(0, 0)$, so we need to find an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a > b$. We know that the length of the major axis, $2a$, is longer than the length of the minor axis, $2b$. So the length of the room, 96, is represented by the major axis and the width of the room, 46, is represented by the minor axis.

Solving for a, we have $2a = 96$, so $a = 48$ and $a^2 = 2304$.

Solving for b, we have $2b = 46$, so $b = 23$ and $b^2 = 529$.

Therefore, the equation of the ellipse is

$$\frac{x^2}{2304} + \frac{y^2}{529} = 1$$

To find the distance between the senators, we must find the distance between the foci, $(\pm c, 0)$, where $c^2 = a^2 - b^2$.

Solving for c, we have

$$\begin{aligned}
c^2 &= a^2 - b^2 \\
c^2 &= 2304 - 529 \\
c &= \pm\sqrt{2304 - 529} \\
c &= \pm\sqrt{1775} \\
c &\approx \pm 42
\end{aligned}$$

The points $(\pm 42, 0)$ represent the foci. Thus the distance between the senators is $2(42) = 84$ feet.