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Centre for Education in
Mathematics and Computing

Grade 9/10 Math Circles

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Triangles - Why Three Angles Are Better Than One

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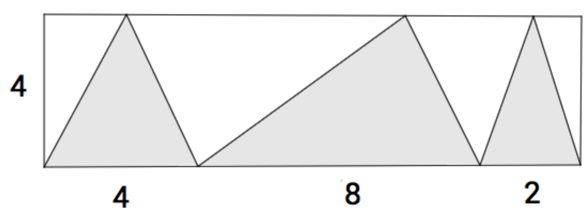
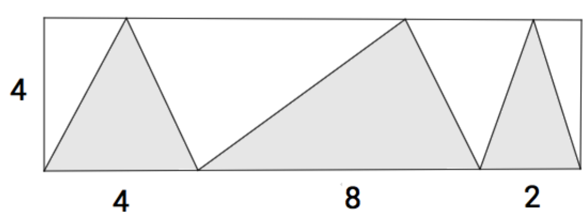
Warm Up Problem

In the rectangle below, determine the area of the shaded region.



- How does the answer change if I change the numbers below the rectangle?
- Does the answer change if I include more triangles?

Solution:



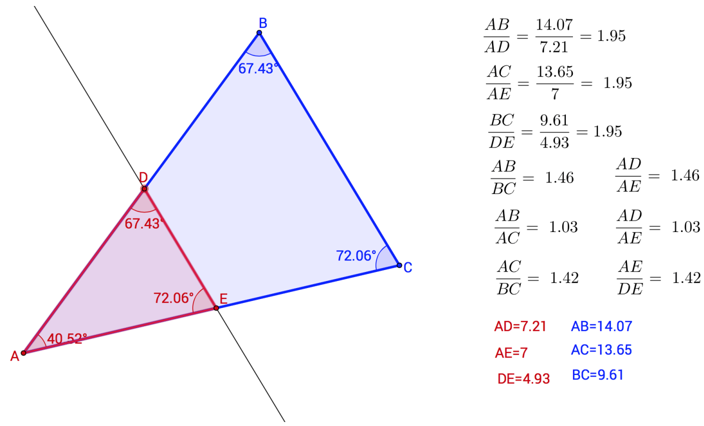
$$
\begin{aligned}
A &= \frac{1}{2}(4)(4) + \frac{1}{2}(4)(8) + \frac{1}{2}(4)(2) \\
&= \frac{4}{2}(4 + 8 + 2) \\
&= (2)(14) \\
&= 28
\end{aligned}
$$

Similar Triangles

Theorem

Triangles with equal angles have side lengths in proportion with each other.

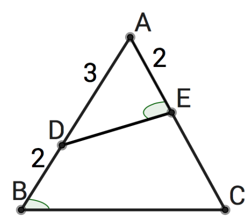
`https://tube.geogebra.org/material/show/id/2547097`



Exercise 1

In the diagram, $\angle ABC = \angle AED$, $AD = 3$, $DB = 2$ and $AE = 2$. Determine the length of EC.

Solution:



Proof

- Look at ΔAED and ΔABC.
- Since $\angle DAE = \angle BAC$ and $\angle ABC = \angle AED$, we have that ΔAED is similar to ΔABC.
- Thus, $\frac{DA}{AE} = \frac{CA}{AB}$. By cross multiplying,

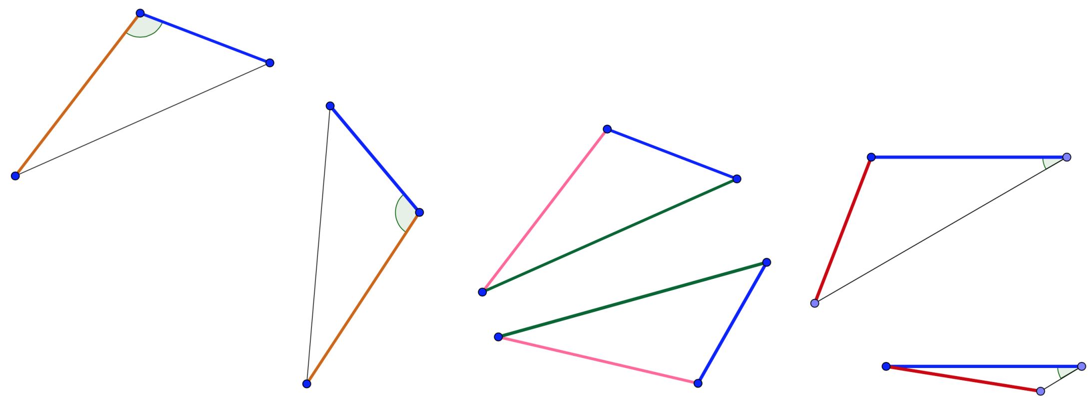
$$CA = \frac{DA \cdot AB}{AE} = \frac{3}{2} \cdot (3 + 2) = \frac{15}{2}.$$

- Hence, $EC = CA - AE = \frac{15}{2} - 2 = \frac{11}{2}$.

Congruent Triangles

Theorem

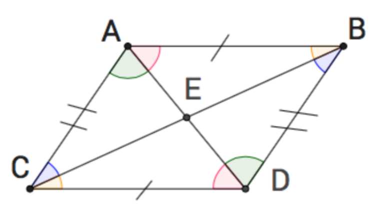
Triangles with equal side lengths OR triangles with two equal sides and equal contained angles are congruent (the same up to translation and rotation).



Careful! the triangles on the right have two equal side lengths and an equal angle but are not congruent (since the angle is not contained).

Theorem

Prove that diagonals of a parallelogram bisect each other.



Proof

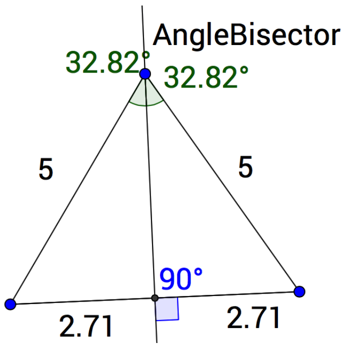
- By the Alternating Angle Theorem (Z pattern), we see that all the coloured angles are equal.
- Since ΔAEC and ΔDEB have the same angles and one equal side, we see that the two triangles are congruent.
- Hence, $AE = DE$ and $EC = EB$.

Note that a square, rectangle and rhombus have the same property.

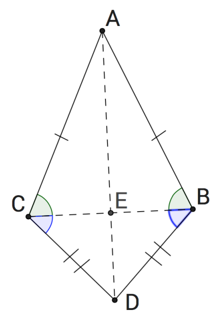
Constructing a Perpendicular Bisector

Theorem

In an isosceles (or equilateral) triangle, the median, perpendicular bisector, and the angle bisector of the angle between the two equal sides are all equal.



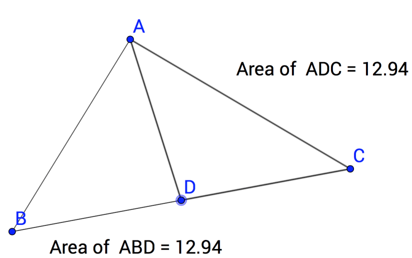
Proof

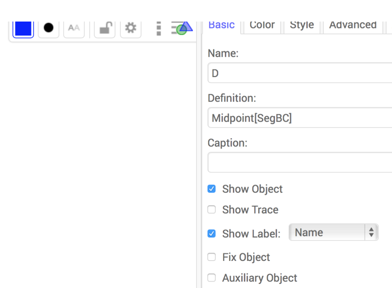


- Since triangles ABC and BCD are isosceles, the coloured angles are equal.
- Now, since $AC = AB$, $CD = BD$ and $\angle ACD = \angle ABD$, we see that ΔACD and ΔABD are congruent.
- Hence $\angle CAE = \angle BAE$, $\angle CDA = \angle BDA$ and thus AD is a bisector of $\angle CAB$.
- Hence AD is a perpendicular bisector for ΔABC and thus the diagonals are perpendicular.

  Note that a rhombus has the same property.

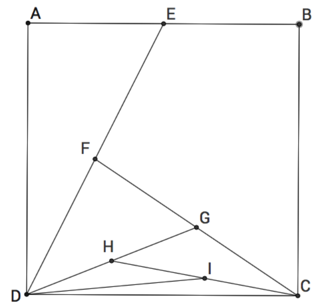
Medians Cut Triangle Area in Half





Exercise 2

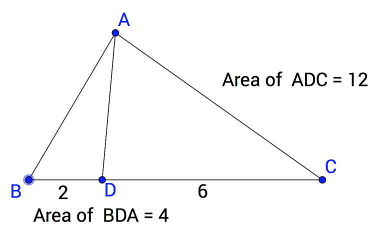
Square $ABCD$ has an area of 4. The point E is the midpoint of AB. Similarly, F, G, H and I are the midpoints of DE, CF, DG and CH respectively. What is the area of triangle IDC?



Triangles: Same Height, Proportional Bases

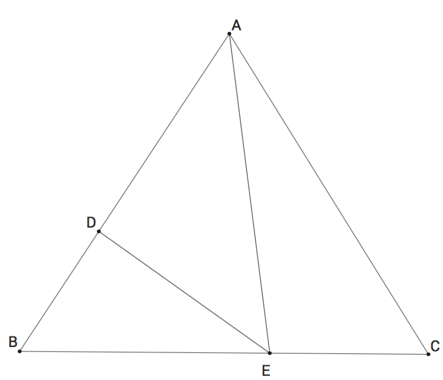
Theorem

If two triangles have the same height, then their areas are proportional to the lengths of their bases.



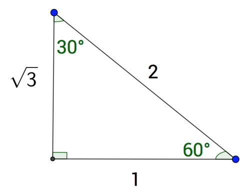
Exercise 3:

In triangle ABC, point D is on AB such that AD is twice as long as DB and E is a point on BC such that BE is twice as long as BC. If the area of triangle ABC is 90 units squared, what is the area of triangle ADE in units squared?

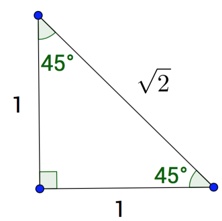


Magic Triangles

$30° : 60° : 90°$ triangles have sides in a $1 : \sqrt{3} : 2$ ratio.

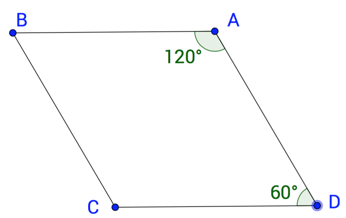


$45° : 45° : 90°$ triangles have sides in a $1 : 1 : \sqrt{2}$ ratio.

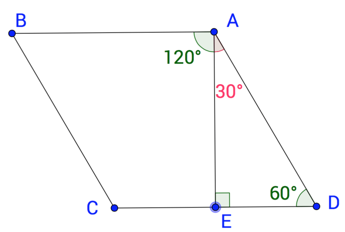


Exercise 4:

Determine the area of the rhombus $ABCD$ with side length 4 and angles shown.



Solution:



Proof

- Draw the perpendicular height from A meeting CD at E.
- This forms a magic triangle. Since $AD = 4$, we know that $ED = 2$ and $AE = 2\sqrt{3}$.
- Hence, the area is $CD \cdot AE = 4 \cdot (2\sqrt{3}) = 8\sqrt{3}$.

Try some of the remaining problems!

- Thanks to Geogebra for all the pictures (Free online!) `http://www.geogebra.org/cms/`
- Check out Euclid the Game online! `http://euclidthegame.com`
- References:
  - Problems, Problems Problems Volumes 7 and 8. `http://cemc.math.uwaterloo.ca/books.html`
  - More of All the Best from the Australian Mathematics Competition (O'Halloran, Pollard, Taylor)