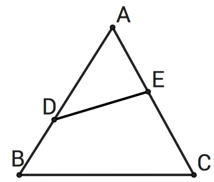
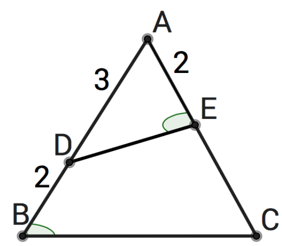
1) **Exercise 1**
In the diagram, $\angle ABC = \angle AED$, $AD = 3$, $DB = 2$ and $AE = 2$. Determine the length of $EC$.




**Solution:**

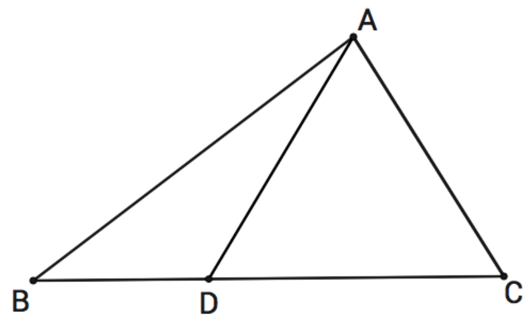



First, we show that $\Delta AED$ and $\Delta ABC$ are similar. Since $\angle DAE = \angle BAC$ and $\angle ABC = \angle AED$, we have that $\Delta AED$ is similar to $\Delta ABC$. Thus, $\frac{DA}{AE} = \frac{CA}{AB}$. By cross multiplying,

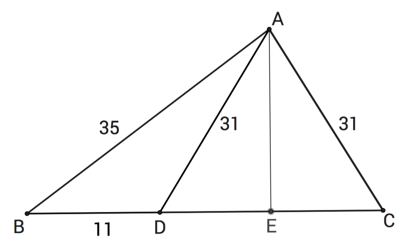
$$CA = \frac{DA \cdot AB}{AE} = \frac{3}{2} \cdot (3+2) = \frac{15}{2}.$$

Hence, $EC = CA - AE = \frac{15}{2} - 2 = \frac{11}{2}$.

2) In triangle $ABC$, $D$ is a point on $BC$. Further, $AB = 35$, $BD = 11$ and $AD = AC = 31$. Determine the length of $DC$.




**Solution:** Construct the Perpendicular as shown




Now, by the Pythagorean Theorem on $\Delta ABE$, we see that

$$\begin{aligned}(AB)^2 &= (BE)^2 + (AE)^2 \\ 35^2 &= (11 + DE)^2 + (AE)^2 \\ 35^2 &= 11^2 + 22 \cdot DE + (DE)^2 + (AE)^2\end{aligned}$$

Now, by the Pythagorean Theorem on $\Delta ADE$, we see that

$$\begin{aligned}(AD)^2 &= (DE)^2 + (AE)^2 \\ 31^2 &= (DE)^2 + (AE)^2\end{aligned}$$

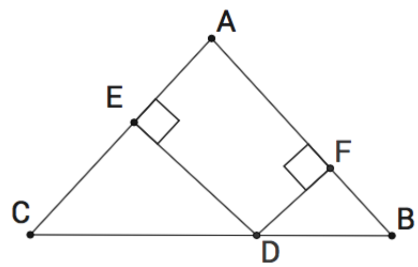
Subtracting these two equations gives

$$35^2 - 31^2 = 11^2 + 22 \cdot DE$$

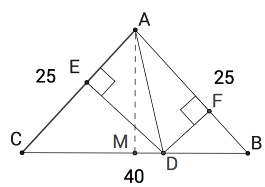
Either using a calculator of factoring as follows, we see that

$$\begin{aligned}35^2 - 31^2 &= 11^2 + 22 \cdot DE \\ (35 - 31)(35 + 31) &= 11(11 + 2 \cdot DE) \\ (4)(66) &= 11(11 + 2 \cdot DE) \\ (4)(6) &= 11 + 2 \cdot DE \\ 24 - 11 &= 2 \cdot DE \\ 13 &= DC && \text{Since } 2DE = DC\end{aligned}$$

3) In triangle $ABC$, $AC = AB = 25$ and $BC = 40$. Point $D$ is a point chosen on $BC$. From $D$, perpendiculars are drawn to meet $AC$ at $E$ and $AB$ at $F$. Determine the value of $DE + DF$.




**Solution:** As shown in the diagram, draw the perpendicular bisector of CB which intersects A and M, the midpoint of BC. Also connect $A$ and $D$



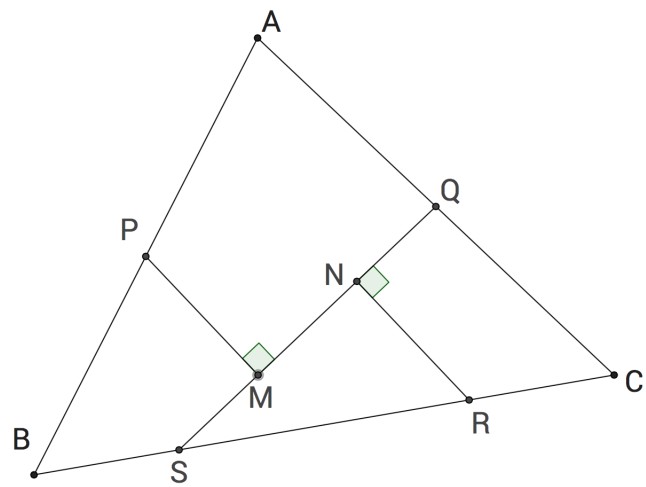

Thus, $MC = 20$ and hence by the Pythagorean Theorem,

$$\begin{aligned}(AC)^2 &= (CM)^2 + (AM)^2 \\ 25^2 - 20^2 &= (AM)^2 \\ (25-20)(25+20) &= (AM)^2 \\ 5 \cdot 45 &= (AM)^2 \\ 5^2 \cdot 9^2 &= (AM)^2 \\ 15 &= AM\end{aligned}$$

Hence, the area of $\Delta ABC$ is $\frac{AM \cdot BC}{2} = 300$. Now, denoting area by absolute values, we see that

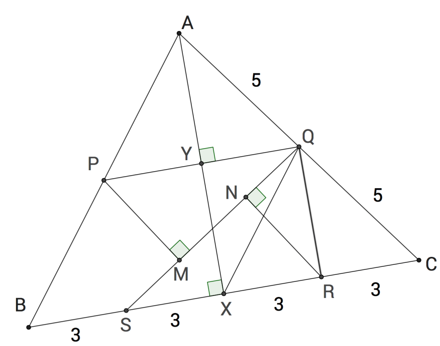
$$\begin{aligned}300 = |\Delta ABC| &= |\Delta ADC| + |\Delta ADB| \\ 300 &= \frac{AC \cdot ED}{2} + \frac{AB \cdot FD}{2} \\ 300 &= \frac{25 \cdot ED}{2} + \frac{25 \cdot FD}{2} \\ 600 &= 25(ED + FD) \\ 24 &= ED + FD\end{aligned}$$

4) Triangle $ABC$ is an isosceles triangle in which $AB = AC = 10$ and $BC = 12$. The points $S$ and $R$ are on $BC$ such that $BS : SR : RB = 1 : 2 : 1$. The midpoints of $AB$ and $AC$ are $P$ and $Q$ respectively. Perpendiculars are drawn from $P$ and $R$ to meet $SQ$ at $M$ and $N$ respectively. What is the length of $MN$?




**Solution:** From the problem statement, $P$ and $Q$ are midpoints hence $AP = PB = AQ = AC = 5$. Let $X$ be the midpoint of BC. Since the triangle $ABC$ is isosceles, we see that $AM$ is the perpendicular bisector of $BC$. Similarly, $AY$ is the perpendicular bisector for $\Delta APQ$. Since the sides are in a $1 : 2 : 1$ ratio, we see that $BS = SX = XR = RC = 3$.

First, we claim that $QR \perp BC$. Draw the triangle below as follows.




Then $\Delta AQY$ is similar to $\Delta ACM$ (They share $\angle CAX = \angle QAY$ and both have a right angle). Hence

$$\frac{AQ}{AY} = \frac{AC}{XC} \qquad \text{implying} \qquad \frac{5}{AY} = \frac{10}{8}$$

Hence $AY = 4$. Similarly, $AQ = 3$. By the Pythagorean Theorem in $\Delta AMC$, we see that

$$\begin{aligned}(AC)^2 &= (AX)^2 + (MC)^2 \\ (10)^2 &= (AX)^2 + 6^2 \\ 100 - 36 &= (AX)^2 \\ 8 &= AX\end{aligned}$$

Hence, $XY = 4$ and by the Pythagorean Theorem again, we see that $QX = 5$. Hence, $\Delta QMC$ is isosceles and thus, $QR$ must be perpendicular since it is the bisector of $CX$.

Now, we can argue as above to show that $PS$ is perpendicular to $BC$. By the Pythagorean Theorem on $\Delta QRC$, we see that

$$\begin{aligned} (QC)^2 &= (QR)^2 + (RC)^2 \\ (5)^2 &= (QR)^2 + 3^2 \\ 25 - 9 &= (QR)^2 \\ 4 &= QR \end{aligned}$$

and similarly, $PS = 4$. Now, as $PS$ and $QR$ are parallel, we see that $\angle PSM = \angle SQR$. Hence $\Delta PSM$ is congruent to $\Delta RQN$ (angles are equal and they share a side length size). Thus $SM = NQ$. Since $\angle QRS$ is a right angle, we see that $\angle SRN = \angle NQR$. Further, $\angle RSN = \angle QRN$. Hence $\Delta RNQ$ which is similar to $\Delta SRQ$. This gives

$$\frac{QN}{QR} = \frac{QR}{QS}$$

Thus, $QN \cdot QS = 16$. Now, by Pythagorean Theorem again on $\Delta QRS$, we see that

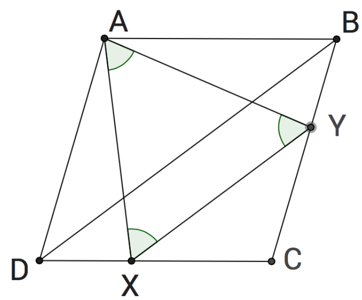
$$\begin{aligned} (QS)^2 &= (QR)^2 + (SR)^2 \\ (QS)^2 &= 4^2 + 6^2 \\ (QS)^2 &= 16 + 36 \\ QS &= \sqrt{52} \\ QS &= 2\sqrt{13} \end{aligned}$$

Thus, $QN = \frac{16}{2\sqrt{13}} = \frac{8}{\sqrt{13}}$. Now, $QS = SM + MN + NQ = 2QN + MN$ and so

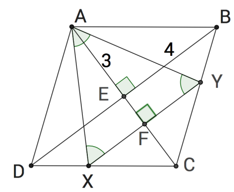
$$MN = QS - 2QN = 2\sqrt{13} - 2 \cdot \frac{8}{\sqrt{13}} = \frac{26 - 16}{\sqrt{13}} = \frac{10}{\sqrt{13}}$$

completing the problem.

5) The lengths of the diagonals $AD$ and $BC$ in rhombus $ABCD$ are 6 and 8 respectively. Triangle $AXY$ is equilateral and line $XY$ is parallel to diagonal $BC$. Determine the length of the altitude of triangle $AXY$.




**Solution:** Construct the picture as shown:




Our goal is to find the length of $AF$. Lines AC and BD are the diagonals. Since the diagonals of a rhombus bisect each other, we see that $EC = AE = 6/2 = 3$ and $DE = BE = 8/2 = 4$. Note also that diagonals of a rhombus meet at right angles. Now, since $DB$ and $XY$ are parallel, $\angle AFY = \angle AEB = 90°$. Thus, since $\angle FYA = 60°$, we see that $\angle YAF = 30°$. Thus, $\Delta YAF$ is a magic triangle and so

$$\frac{AF}{FY} = \frac{\sqrt{3}}{1}$$

Giving $AF = \sqrt{3} \cdot FY$ and so $AF = 3 + EF = \sqrt{3} \cdot FY$. Next, we note that $\Delta CEB$ and $\Delta CFY$ are similar since they share $\angle YCF = \angle BCE$ and $\angle CEB = \angle CFY$. Thus, by similar triangles, we see that

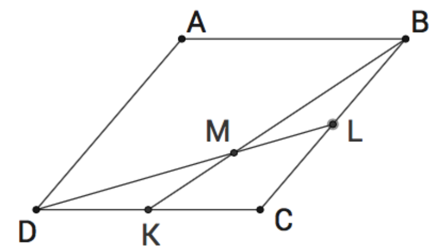
$$\frac{FC}{FY} = \frac{CE}{BE} = \frac{3}{4}.$$

Hence $4FC = 3FY$. Now, $FC = 3 - EF$ and thus, $12 - 4EF = 3FY$. Solving for $FY$ by substituting $EF = \sqrt{3} \cdot FY - 3$ gives

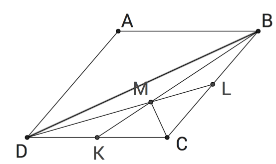
$$\begin{aligned}
12 - 4(\sqrt{3} \cdot FY - 3) &= 3FY \\
12 - 4\sqrt{3} \cdot FY + 12 &= 3FY \\
24 &= 3FY + 4\sqrt{3}FY \\
24 &= (3 + 4\sqrt{3})FY \\
\frac{24}{3 + 4\sqrt{3}} &= FY
\end{aligned}$$

Thus, $AF = \sqrt{3} \cdot FY = \frac{24\sqrt{3}}{3+4\sqrt{3}}$ completing the question.

6) In the diagram, $ABCD$ is a rhombus with $K$ the midpoint of $DC$ and $L$ the midpoint of $BC$. Segments $DL$ and $BK$ intersect at $M$. Determine the fraction of the area of quadrilateral $KMLC$ is of the area of the rhombus $ABCD$.




**Solution:** Denote areas by absolute values. Join the diagonal DB and points $MC$ as is done in the following diagram




Now, Since $K$ is the midpoint of $DC$, we see that $|\Delta MDK| = |\Delta MKC|$. Similarly, $|\Delta MLC| = |\Delta MLB|$. Looking at $\Delta DBC$, we see that $\Delta BDK| = |\Delta BKC|$. Hence, we have that

$$|\Delta BDM| + |\Delta MDK| = |KMLC| + |\Delta BML|$$

Similarly with $\Delta DBL$ and $\Delta DLC$, we see that

$$|\Delta BDM| + |\Delta BML| = |KMLC| + |\Delta MDK|$$

Subtracting these two gives

$$|\Delta MDK| - |\Delta BML| = |\Delta BML| - |\Delta MDK|$$

Which gives that $2|\Delta BML| = 2|\Delta MDK|$ and hence $|\Delta BML| = |\Delta MDK|$. This gives us that

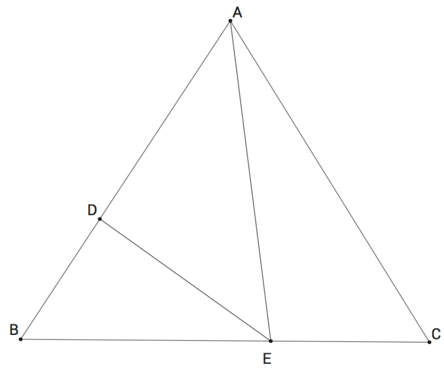
$$|KMLC| = |\Delta MKC| + |\Delta MLC| = |\Delta MDK| + |\Delta BML| = 2|\Delta MDK|$$

Thus, in the above, we see that $\Delta BDM| = |KMLC| = 2|\Delta MDK|$. Since $\Delta DBC$ is half the rhombus, we see that

$$\begin{aligned}|ABCD|/2 = |\Delta DBC| &= |\Delta BDM| + |KMLC| + |\Delta MDK| + |\Delta BLM| \\ &= 6|\Delta MDK| \\ &= 3|KMLC|\end{aligned}$$

and hence $KMLC$ is one sixth of the area of the rhombus.

7) **Exercise 3:**
In triangle $ABC$, point $D$ is on $AB$ such that $AD$ is twice as long as $DB$ and $E$ is a point on $BC$ such that $BE$ is twice as long as $BC$. If the area of triangle $ABC$ is 90 units squared, what is the area of triangle $ADE$ in units squared?




**Solution:** From the problem statement, we see that $BE = 2BC$. Note that $\Delta ABE$ and $\Delta AEC$ have the same heights and so their areas are in proportion with their bases. Thus, denoting area by absolute values, $|\Delta ABE| = 2|\Delta AEC|$. The problem also tells us that (suppressing units throughout)

$$90 = |\Delta ABC| = |\Delta ABE| + |\Delta AEC| = 3|\Delta AEC|$$

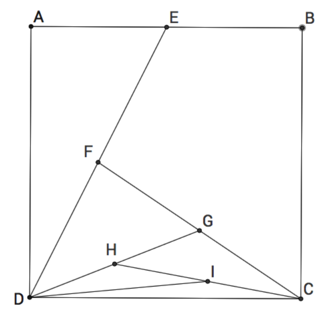
and hence $|\Delta AEC| = 30$. Also from the problem statement, we see that $AD = 2DB$. Note that $\Delta AED$ and $\Delta DEB$ have the same heights and so their areas are in proportion with their bases. Thus, denoting area by absolute values, $|\Delta AED| = 2|\Delta DEB|$. Since

$$60 = 2|\Delta AEC| = |\Delta ABE| = |\Delta AED| + |\Delta DEB| = 3|\Delta DEB|$$

we see that $|\Delta DEB| = 20$ and hence $|\Delta AED| = 2|\Delta DEB| = 40$.

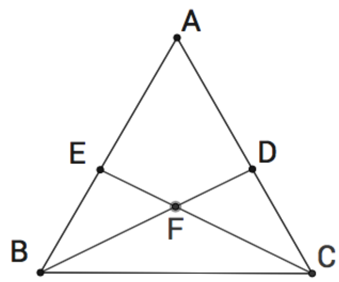
8) **Exercise 2**

Square $ABCD$ has an area of 4. The point $E$ is the midpoint of $AB$. Similarly, $F, G, H$ and $I$ are the midpoints of $DE$, $CF$, $DG$ and $CH$ respectively. What is the area of triangle $IDC$?

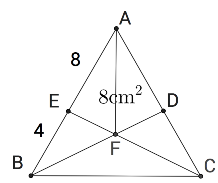



**Solution:** As usual, denote area by absolute values. Connect $EC$ to form an isosceles triangle $DEC$. Now, since $F$ is the midpoint of $DE$, we see that $|\Delta DFC| = 0.5|\Delta DEC|$ since the median cuts the area in half. Similarly, $|\Delta DGC| = 0.5|\Delta DFC|$, . $|\Delta DHC| = 0.5|\Delta DGC|$, $|\Delta DIC| = 0.5|\Delta DHC|$. Combining this gives $|\Delta DIC| = (0.5)^4|\Delta DEC| = \frac{1}{16} \cdot |\Delta DEC|$. Since the height of $\Delta DEC$ is 2 and the base is 2, it's area is 2 and hence $\Delta DIC| = \frac{2}{16} = \frac{1}{8}$.

9) In the diagram, $AB = AC = 12\text{cm}$ and $AE = AD = 8\text{cm}$. The area of quadrilateral $AEFD$ is $8\text{cm}^2$. What is the area of triangle $ABC$ in square centimetres?

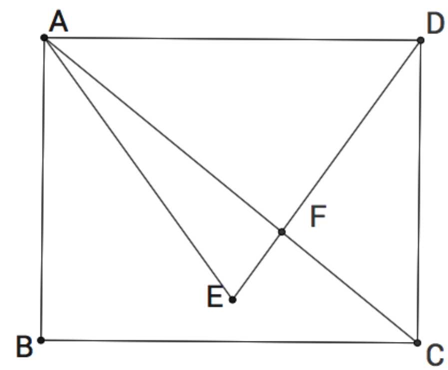



**Solution:** We suppress units until the end of this argument. Construct segment AF as shown.

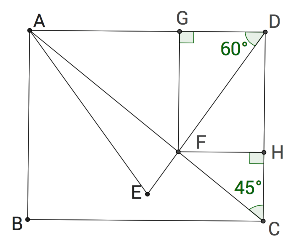



Now, $\Delta AEF$ and $\Delta EFB$ have the same height and their bases are in a $2:1$ proportion with each other. Further, $AF$ cuts the area of quadrilateral $AEFD$ in half by symmetry so $\Delta AEF$ has area 4. Hence, the area of $\Delta EFB$ is half the area of $\Delta AEF$ which gives the area of $\Delta EFB$ to be 2. By symmetry, $\Delta DFC$ also has area 2. Lastly, using $\Delta AEC$ which has area $8 + 2 = 10$ (adding the area of quadrilateral to the triangle $DFC$) and using $\Delta ECB$ which has area $2 + |\Delta FCB|$ (here the absolute value signs denote area), we see that since the heights of these triangles are the same and the side lengths are in a $2:1$ ration with each other, we have that $2 + |\Delta FCB| = \frac{10}{2}$ and thus $|\Delta FCB| = 3$. Adding the three triangle areas and the quadrilateral area gives $8+2+2+3 = 15$ square centimetres as the area.

10) The diagram shows a square $ABCD$ with unit length. Triangle $ADE$ is equilateral. The diagonal $AC$ of square $ABCD$ intersects line segment $DE$ at the point $F$. What is the area of triangle $AFD$?




**Solution:** Begin by constructing the following line segments on the diagram

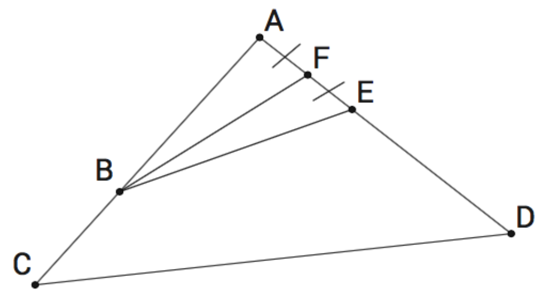



Our goal is to find $|\Delta FDC|$, the area of $\Delta FDC$. This is given by $|\Delta FDC| = \frac{1}{2}FH \cdot DC = \frac{FH}{2}$. Now, $FGDH$ is a rectangle and hence $GD = FH$. Since $AC$ is a diagonal of a square, $\angle HCF = 45°$. Since $\Delta FHC$ is a magic triangle, we see that $GD = FH = HC$. Thus, $DH = DC - HC = 1 - GD$. Since $FGDH$ is a rectangle, once again we have that $FG = DH = 1 - GD$. Since $\Delta FGD$ is a $30° : 60° : 90°$ triangle, we see that

$$\begin{aligned} \frac{GD}{FG} &= \frac{1}{\sqrt{3}} \\ \frac{GD}{1 - GD} &= \frac{1}{\sqrt{3}} \\ \sqrt{3} \cdot GD &= 1 - GD \\ (\sqrt{3} + 1)GD &= 1 \end{aligned}$$

Solving gives $GD = \frac{1}{\sqrt{3}+1}$ or $GD = \frac{\sqrt{3}-1}{2}$. Since $GD = FH$, we see that $|\Delta FDC| = \frac{FH}{2} = \frac{\sqrt{3}-1}{4}$ completing the question.

11) Below, $F$ is the midpoint of $AE$, $AE = \frac{1}{2}ED$, $AB = 9$ and $BC = 3$. If the area of quadrilateral $BEDC$ is 72, then what is the area of triangle $BFE$?




**Solution:** Join $BD$. Denoting area by absolute values, we see that $|\Delta BFE| = |\Delta ABF|$ since the triangles have the same base and height. Since $DE = 2AE$, we see that

$$|\Delta BED| = 2|\Delta BAE| = 2(|\Delta BFE| + |\Delta ABF|) = 2(|\Delta BFE| + |\Delta BFE|) = 4|\Delta BFE|$$

Now, Since $AB : BC$ is $3 : 1$, we see that $|\Delta ABD| = 3|\Delta BCD|$. Since

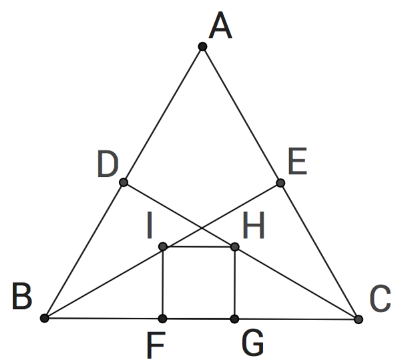
$$|\Delta ABD| = |\Delta BFE|+|\Delta ABF|+|\Delta BED| = |\Delta BFE|+|\Delta BFE|+4|\Delta BFE| = 6|\Delta BFE|$$

Combining gives $|\Delta BCD| = 2|\Delta BFE|$. Thus,

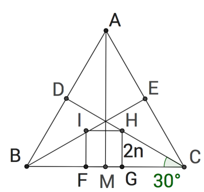
$$72 = |BEDC| = |\Delta BED| + |\Delta BCD| = 4|\Delta BFE| + 2|\Delta BFE| = 6|\Delta BFE|$$

Hence, $|\Delta BFE| = 12$.

12) In the diagram, triangle $ABC$ is equilateral with sides of length 2. Line segments $CD$ and $EB$ are medians and $FGHI$ is a square. Determine the ratio of the area of square $FGHI$ to triangle $ABC$.




**Solution:** Draw the altitude $AM$ which is also the perpendicular bisector of $BC$. Let $MG = FG = n$ so that the square has side length $2n$.




From the problem statement, we know that $MC = BC/2 = 1$. Hence $GC = 1 - n$. Now, triangle $HGC$ is a magic triangle since the angles are $30° : 60° : 90°$. Hence, we see that

$$\frac{2n}{1-n} = \frac{HG}{GC} = \frac{1}{\sqrt{3}}$$

This gives $2\sqrt{3}n = 1 - n$ and so $n = \frac{1}{2\sqrt{3}+1}$. Hence, the area of the square $FGHI$ is $4n^2 = \frac{4}{13+4\sqrt{3}}$. The area of the triangle $ABC$ is $\frac{1}{2} \cdot AM \cdot BC = \frac{1}{2}\sqrt{3} \cdot 2 = \sqrt{3}$ (note that $\Delta ABM$ is another magic triangle) and thus, the ratio of the area of this square to triangle $ABC$ is

$$\frac{\frac{4}{13+4\sqrt{3}}}{\sqrt{3}} = \frac{4}{12 + 13\sqrt{3}}$$