Intermediate Math Circles
Wednesday February 24, 2016
Introduction to Vectors

We write vectors using **vector notation**. This includes

- Write components as column with square brackets.
- Using a directed line segment (ray).
- Use an arrow over letter to tell its a vector.

$$\text{e.g., } \vec{z} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Definition 1: A **vector** must have

- Direction.
- Length.

Definition 2: In 2D, the zero vector has both components equal to 0.

It is denoted as $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and has no length and an undefined direction.

Vector Addition

We can add vectors by adding their components.

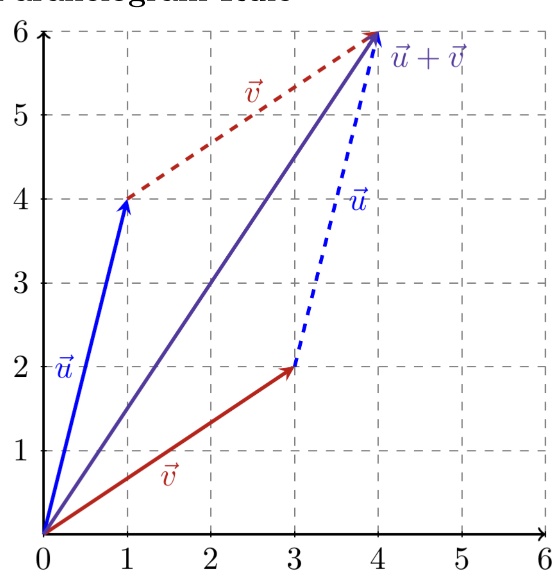
Example:

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+0 \\ 4+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

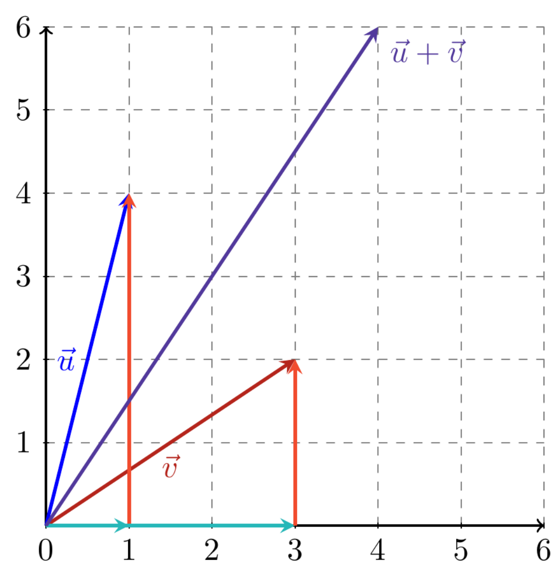
$$\begin{bmatrix} 1/2 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2+3 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 3 \end{bmatrix}$$

Graphing Addition of Vectors

Parallelogram Rule



"Tip to Tail"



Scalar Multiplication

Definition 3: If $\vec{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ and t is a scalar, then scalar multiplication is defined as

$$t\vec{z} = t\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} tz_1 \\ tz_2 \end{bmatrix}$$

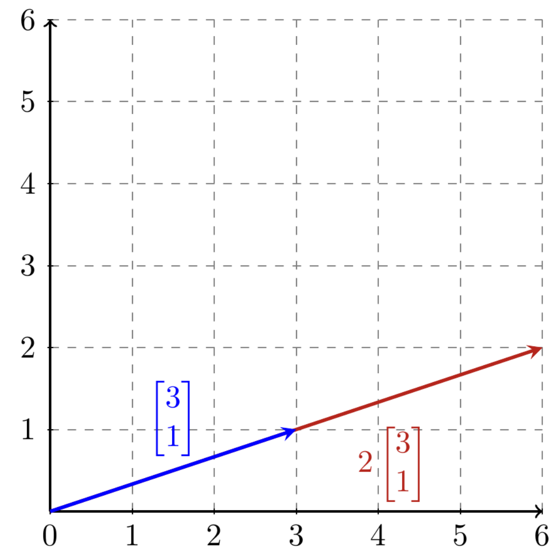
Example:

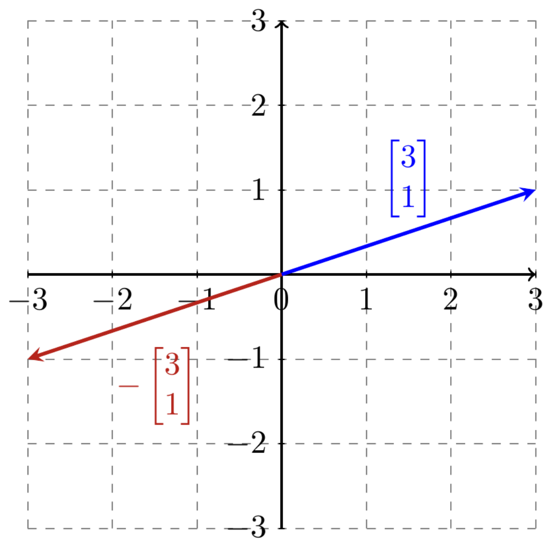
$$2\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \times 3 \\ 2 \times 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

Subtraction is a combination of both addition and scalar multiplication.

Graphing Scalar Multiplication

Scalar multiplication stretches or shrinks a vector.





Vectors and Lines

Consider the vector $\vec{z} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

- How can we turn $\vec{z}$ into a line passing through the origin?

$$\vec{z} = t\begin{bmatrix} 3 \\ 2 \end{bmatrix}, t \in \mathbb{R}$$

- How can we make this line pass through the point $(1, 1)$?

$$\vec{z} = t\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Definition 4: A line through $\vec{p}$ with direction vector $\vec{d}$ has the **vector equation**

$$\vec{z} = \vec{p} + t\vec{d}, t \in \mathbb{R}$$

Question: How can we tell if two vector equations are parallel?

Two vectors are parallel if their direction vectors, $\vec{d}$) are scalar multiples of each other.

Definition 5: The **parametric equation** of the line $\vec{z} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + t\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ is the collection of equations

$$\begin{aligned} z_1 &= p_1 + td_1 \\ z_2 &= p_2 + td_2 \end{aligned}$$

Example: Does the vector equation $\vec{z} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ pass through the point $(0, -1)$?

Putting $\vec{z}$ into parametric form:

$$\begin{aligned} z_1 &= -2 + t(-1) && \implies z_1 = -2 - t \\ z_2 &= 1 + t(1) && \implies z_2 = 1 + t \end{aligned}$$

We want $z_1 = 0$, therefore $0 = -2 - t$ or $t = -2$.
We check this in our second equation:

$$z_2 = 1 + (-2) = -1$$

as required.

For a parametric equation, the value of t must be the same for every equation!
Example: Solve for the unknown variables.

$$m\begin{bmatrix} 1/2 \\ 3 \end{bmatrix} + n\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 9 \end{bmatrix}$$

We split this into its parametric equations:

$$\begin{aligned} m(1/2) + n(-3) &= 11 && \implies \tfrac{m}{2} - 3n = 11 && (1) \\ m(3) + n(1) &= 9 && \implies 3m + n = 9 && (2) \end{aligned}$$

Isolate for n in equation (2): $n = 9 - 3m$

Substitute this into equation (1): $\frac{m}{2} - 3(9 - 3m) = 11$

Simplifying, we have,

$$\frac{m}{2} - 27 + 9m = 11$$
$$9.5m = 38$$
$$m = 4$$

Substitute $m = 4$ into (2):

$3(4) + n = 9$

Therefore, $n = -3$.

Exploring another vector form — Scalar Form

Consider $\vec{z} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Put into parametric form:

$$z_1 = -2 - t$$
$$z_2 = 1 + t$$

In each case, solve for t.

$$t = -2 - z_1 \quad (1) \qquad\qquad t = -1 + z_2 \quad (2)$$

Since these are from the same vector equation, $t = t$:

$$(1) = (2)$$
$$-2 - z_1 = -1 + z_2$$

Solve for z_2:

$$z_2 = -z_1 - 1$$