Intermediate Math Circles
Wednesday February 24, 2016
Problem Set 1

1. Let $\vec{u} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$. Find the following:

   (a) $\vec{u} + \vec{v} + \vec{w}$

   (b) $3\vec{u} - 2\vec{v}$

   (c) $-2\vec{u} + \frac{1}{8}\vec{v} + 3\vec{w}$

2. Simplify

$$(a+b+c)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-a+b)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-a+b+c)\begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

3. Find a and b.

   (a) $\begin{bmatrix} a \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ a+b \end{bmatrix}$

   (b) $\begin{bmatrix} 2 \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

   (c) $3a\begin{bmatrix} 1 \\ b \end{bmatrix} = b\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

4. Find the unknown variables:

$$m\begin{bmatrix} 1 \\ 2 \end{bmatrix} + n\begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

5. Sometimes a pair of vectors can be added or subtracted to become *any* other vector.

   (a) Solve for m and n in terms of x and y:

   $$m\begin{bmatrix}1\\2\end{bmatrix}+n\begin{bmatrix}-2\\-1\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$$

   (b) Let $x = 3$ and $y = 3$, what do m and n equal?

   (c) Demonstrate this graphically. The implication is that any vector $\begin{bmatrix}x\\y\end{bmatrix}$ can be represented as some combination of $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}-2\\-1\end{bmatrix}$.

   (d) Choose your favourite numbers to be x and y. Solve for m and n.

   (e) Solve for p and q in terms of x and y:

   $$p\begin{bmatrix}1\\1\end{bmatrix}+q\begin{bmatrix}-1\\-1\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$$

   Can any vector be represented by these two vectors? Demonstrate graphically.
   2em

6. Let $x_1 = p_1 + td_1$ and $x_2 = p_2 + td_2$, $t \in \mathbb{R}$. Following the scalar form of the line example, solve for x_2 in terms of x_1.

Investigation

Using what you've learned, work your way through the following questions to learn about the norm of a vector.

The **norm** of a vector, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, is defined as $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2}$.

Notice that this will always be a scalar (a number) rather than a vector.

1. Find the norm of the following vectors: (a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 4 \end{bmatrix}$ (c) $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ (d) $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$

2. Plot each vector. What do you think the norm represents?

3. Does the equation for the norm look similar to another familiar equation? This is not a coincidence. Justify why this similarity makes sense.

4. Is it possible for $\|\vec{x}\| < 0$? Justify your answer.

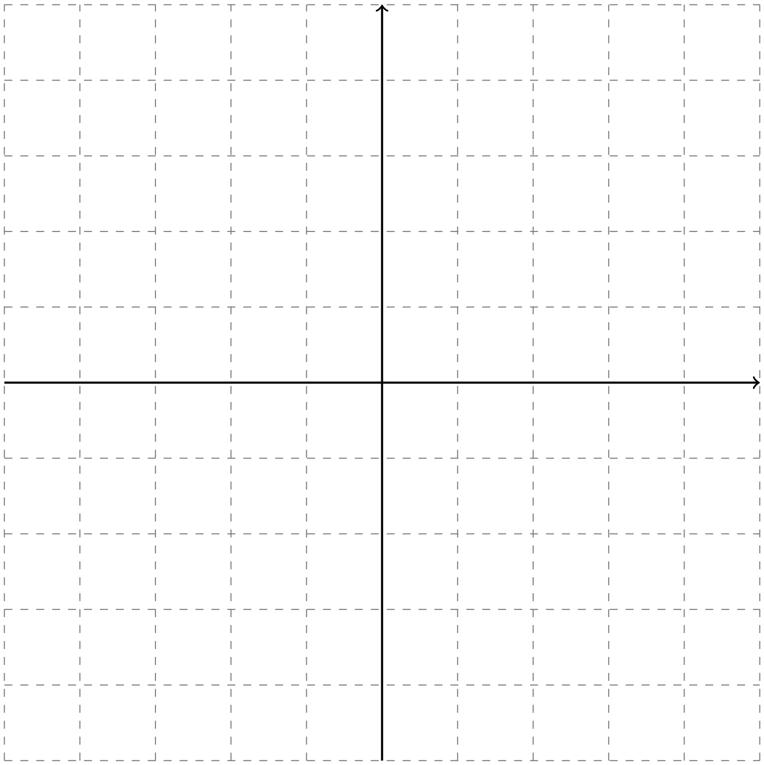
5. **Normalizing** a vector, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, is defined by the operation $\hat{x} = \dfrac{\vec{x}}{\|\vec{x}\|}$. That is, we take a vector and divide each component of the vector by the norm.

   We use a 'hat' ($\hat{}$) on the vector to tell that a vector has been normalized.

   (a) Normalize $\vec{u} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

   (b) Now find the norm of $\hat{u}$ and $\hat{v}$. What do you notice?

(c) Plot $\vec{v}$ and $\hat{v}$ on the same graph.



(d) What does normalizing a vector do to the vector? Why might someone want to do this? If you're not sure, consider that a normalized vector is also known as a **unit** vector.

Sudoku to end the day

2	5			3		9		1
	1				4			
4		7				2		8
		5	2					
				9	8	1		
	4				3			
			3	6			7	2
	7							3
9		3				6		4