Intermediate Math Circles
Wednesday February 24, 2016
Problem Set 1 — Solutions

1. Let $\vec{u} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$. Find the following:

   (a) $\vec{u} + \vec{v} + \vec{w}$

   (b) $3\vec{u} - 2\vec{v}$

   (c) $-2\vec{u} + \frac{1}{8}\vec{v} + 3\vec{w}$

   Solution

   (a) $\vec{u} + \vec{v} + \vec{w} = \begin{bmatrix} 3 \\ -7 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \end{bmatrix}$

   (b) $3\vec{u} - 2\vec{v} = 3\begin{bmatrix} 3 \\ -7 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ -27 \end{bmatrix} + \begin{bmatrix} 0 \\ -8 \end{bmatrix} = \begin{bmatrix} 9 \\ -35 \end{bmatrix}$

   (c) $-2\vec{u} + \frac{1}{8}\vec{v} + 3\vec{w} = -2\begin{bmatrix} 3 \\ -7 \end{bmatrix} + \frac{1}{8}\begin{bmatrix} 0 \\ 4 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} -6 \\ 14 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$

2. Simplify

$$(a+b+c)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-a+b)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-a+b+c)\begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

   Solution

$$\begin{aligned} & a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \end{bmatrix} - a\begin{bmatrix} 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \end{bmatrix} - a\begin{bmatrix} -1 \\ -1 \end{bmatrix} + b\begin{bmatrix} -1 \\ -1 \end{bmatrix} + c\begin{bmatrix} -1 \\ -1 \end{bmatrix} \\ &= a\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) + b\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) + c\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) \\ &= a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

3. Find a and b.

   (a) $\begin{bmatrix} a \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ a+b \end{bmatrix}$

   (b) $\begin{bmatrix} 2 \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

   (c) $3a\begin{bmatrix} 1 \\ b \end{bmatrix} = b\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

   Solution

(a) $\begin{bmatrix} a \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ a+b \end{bmatrix}$

We have $a = 4$. Thus, $6 = a + b \implies 6 = 4 + b$. Therefore, $b = 2$.

(b) $\begin{bmatrix} 2 \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

We have $2 = a$. Thus, $b = 3a \implies b = 3(2) = 6$.

(c) $3a \begin{bmatrix} 1 \\ b \end{bmatrix} = b \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Putting in parametric form

$$\begin{aligned} 3a &= 2b \\ 3ab &= b \end{aligned}$$

Thus, $a = \frac{2}{3}b$. Solving, we have

$$\begin{aligned} 3\left(\frac{2}{3}b\right)b &= b \\ 2b^2 - b &= 0 \\ b(2b-1) &= 0 \\ \therefore b = 0 \text{ or } b &= \frac{1}{2} \end{aligned}$$

If $b = 0$, then $a = 0$. If $b = \frac{1}{2}$, then $a = \frac{1}{3}$.

4. Find the unknown variables:

$$m \begin{bmatrix} 1 \\ 2 \end{bmatrix} + n \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Solution

Putting into parametric form, we have

$$\begin{aligned} m - 2n &= 3 \\ 2m - n &= 3 \end{aligned}$$

Thus $n = 2m - 3$. Therefore, $m - 2(2m - 3) = 3$ or $m = 1$.

Thus, $n = 2(1) - 3 = -1$.

5. Sometimes a pair of vectors can be added or subtracted to become *any* other vector.

(a) Solve for m and n in terms of x and y:

$$m\begin{bmatrix}1\\2\end{bmatrix}+n\begin{bmatrix}-2\\-1\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$$

Solution

Putting into parametric form, we have

$$\begin{aligned}m-2n&=x\\2m-n&=y\end{aligned}$$

Thus $n=2m-y$. Therefore, $m-2(2m-y)=x$ or $m=-\dfrac{x-2y}{3}=-\dfrac{1}{3}x+\dfrac{2}{3}y$.

Thus, $n=2\left(-\dfrac{1}{3}x+\dfrac{2}{3}y\right)-y=-\dfrac{2}{3}x+\dfrac{1}{3}y$.

(b) Let $x=3$ and $y=3$, what do m and n equal?

Solution

Solving, we have

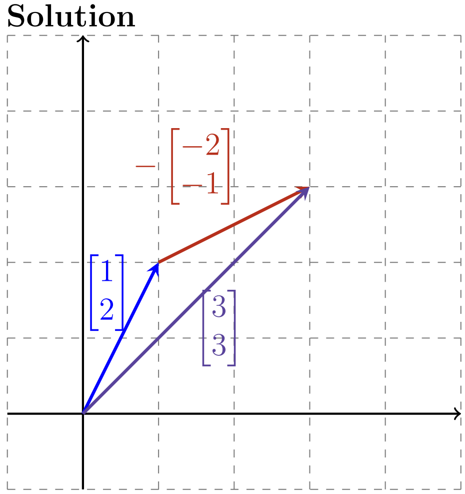
$$m=-\frac{1}{3}(3)+\frac{2}{3}(3)=-1+2=1$$

and

$$n=-\frac{2}{3}(3)+\frac{1}{3}(3)=-2+1=-1$$

(c) Demonstrate this graphically. The implication is that any vector $\begin{bmatrix}x\\y\end{bmatrix}$ can be represented as some combination of $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}-2\\-1\end{bmatrix}$.

Solution



(d) Choose your favourite numbers to be x and y. Solve for m and n.

(e) Solve for p and q in terms of x and y:

$$p\begin{bmatrix}1\\1\end{bmatrix} + q\begin{bmatrix}-1\\-1\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$$

Can any vector be represented by these two vectors? Demonstrate graphically.

Solution

Putting into parametric form, we have

$$\begin{aligned} p - q &= x \\ p - q &= y \end{aligned}$$

Therefore, $x = y$. So only vectors where the first and second components are the same can be represented by these vectors.

6. Let $x_1 = p_1 + td_1$ and $x_2 = p_2 + td_2$, $t \in \mathbb{R}$. Following the scalar form of the line example, solve for x_2 in terms of x_1.

Solution

Solve for t:

$$\begin{aligned} x_1 &= p_1 + td_1 \\ \frac{x_1 - p_1}{d_1} &= t \end{aligned} \qquad\qquad \begin{aligned} x_2 &= p_2 + td_2 \\ \frac{x_2 - p_2}{d_2} &= t \end{aligned}$$

Since these two equations are from the same vector equation, the scalar t must be the same in both cases.

Therefore,

$$\begin{aligned} \frac{x_1 - p_1}{d_1} &= \frac{x_2 - p_2}{d_2} \\ \frac{d_2}{d_1}(x_1 - p_1) &= x_2 - p_2 \\ \frac{d_2}{d_1}(x_1 - p_1) + p_2 &= x_2 \end{aligned}$$

Investigation

Using what you've learned, work your way through the following questions to learn about the norm of a vector.

The **norm** of a vector, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, is defined as $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2}$.

Notice that this will always be a scalar (a number) rather than a vector.

1. Find the norm of the following vectors: (a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 4 \end{bmatrix}$ (c) $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ (d) $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$

   Solution

   (a) $\left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\| = \sqrt{1^2 + 0^2} = 1$

   (b) $\left\| \begin{bmatrix} 0 \\ 4 \end{bmatrix} \right\| = \sqrt{0^2 + 4^2} = 4$

   (c) $\left\| \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

   (d) $\left\| \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

2. Plot each vector. What do you think the norm represents?

   The norm represents the length of the vector.

3. Does the equation for the norm look similar to another familiar equation? This is not a coincidence. Justify why this similarity makes sense.

   The equation is similar to the Pythagorean theorem. This is because when we split a vector into its horizontal and vertical components, we create a right angled triangle. The length of the vector is this triangle's hypotenuse.

4. Is it possible for $\|\vec{x}\| < 0$? Justify your answer.

   Solution

   In the equation for the norm, each component is squared, so it is not possible to have a negative number as a result.

5. **Normalizing** a vector, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, is defined by the operation $\hat{x} = \dfrac{\vec{x}}{\|\vec{x}\|}$. That is, we take a vector and divide each component of the vector by the norm.

   We use a 'hat' ($\hat{}$) on the vector to tell that a vector has been normalized.

   (a) Normalize $\vec{u} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

   Solution $\hat{u} = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ and $\hat{v} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$

(b) Now find the norm of $\hat{u}$ and $\hat{v}$. What do you notice?

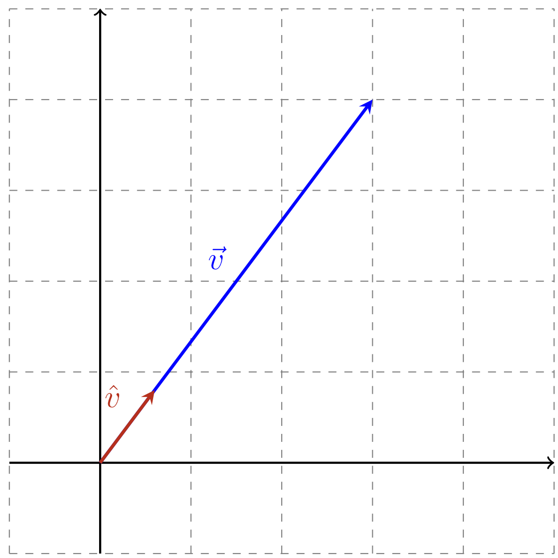
Solution

$$\|\vec{u}\| = \sqrt{\left(\frac{-1}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2} = 1$$

$$\|\vec{v}\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{25}{25}} = 1$$

(c) Plot $\vec{v}$ and $\hat{v}$ on the same graph.

Solution



(d) What does normalizing a vector do to the vector? Why might someone want to do this? If you're not sure, consider that a normalized vector is also known as a **unit** vector.

A normalized vector always has a length of 1.

Sudoku to end the day

2	5	8	7	3	6	9	4	1
6	1	9	8	2	4	3	5	7
4	3	7	9	1	5	2	6	8
3	9	5	2	7	1	4	8	6
7	6	2	4	9	8	1	3	5
8	4	1	6	5	3	7	2	9
1	8	4	2	6	9	5	7	2
5	7	6	1	4	2	8	9	3
9	2	3	5	8	7	6	1	4