Intermediate Math Circles
Wednesday March 02, 2016
Introduction to Vectors II

Review

A **vector** must have a magnitude and a direction.

If $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, then we have the following properties:

1. Vector Addition: $\vec{u} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$

2. Scalar Multiplication: $t\vec{v} = t\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} tv_1 \\ tv_2 \end{bmatrix}, t \in \mathbb{R}$

3. Norm/Length of a Vector: $\|v\| = \sqrt{(v_1)^2 + (v_2)^2}$

4. Unit Vector: $\hat{v} = \frac{1}{\|\vec{v}\|}\vec{v}$

Dot Product

If $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, then the **dot product** is a scalar value defined as

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$$

Example: If $\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, find the dot product of $\vec{u}$ and $\vec{v}$.

$$\begin{aligned} \vec{u} \cdot \vec{v} &= (3)(-2) + (4)(1) \\ &= -6 + 4 \\ &= -2 \end{aligned}$$

Example: Consider $\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

1. What is the length $\vec{u}$?
$$\|\vec{u}\| = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$$

2. What is $\vec{u} \cdot \vec{u}$?
$$\vec{u} \cdot \vec{u} = (3)^2 + (4)^2 = 25$$

3. How does this relate to the length of $\vec{u}$?

DEFINITION: The length of $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}.$$

FACT: We often square this result to get $\|\vec{u}\|^2 = \vec{u} \cdot \vec{u}$

TRY THIS: Find the following dot products. What is special about the two vectors in each case?

1. $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \vec{a} \cdot \vec{b} = (1)(-1) + (1)(1) = 0$
2. $\vec{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{d} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \vec{c} \cdot \vec{d} = (1)(0) + (0)(1) = 0$
3. $\vec{e} = \begin{bmatrix} 6 \\ 8 \end{bmatrix} \quad \vec{f} = \begin{bmatrix} -2 \\ 3/2 \end{bmatrix} \Rightarrow \vec{e} \cdot \vec{f} = (6)(-2) + (8)\left(\frac{3}{2}\right) = 0$
4. $\vec{g} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad \vec{h} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{g} \cdot \vec{h} = (2)(0) + (5)(0) = 0$

DEFINITION: Two vectors, $\vec{u}$ and $\vec{v}$, are <u>orthogonal (perpedicular)</u> if <u>$\vec{u} \cdot \vec{v} = 0$</u>.

FACT: The zero vector ($\vec{0}$) is orthogonal to every other vector.

DEFINITION: If $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and θ is the angle between the two vectors, then

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

DEFINITION: If $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, then

$$d(u, v) = \|u - v\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$$

is the **distance** from $\vec{u}$ to $\vec{v}$.

EXAMPLE: Find the distance between $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

$$d(u, v) = \|\vec{u} - \vec{v}\| = \sqrt{(3 - 4)^2 + (1 - 1)^2} = \sqrt{1} = 1$$

Prove that $\|t\vec{u}\| = |t|\ \|\vec{u}\|$ for any vector $\vec{u}$ and $t \in \mathbb{R}$.

$$\begin{aligned}
\|t\vec{u}\| &= \left\| t \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\| \\
&= \sqrt{(tu_1)^2 + (tu_2)^2} \\
&= \sqrt{t^2(u_1)^2 + t^2(u_2)^2} \\
&= \sqrt{t^2}\sqrt{(u_1)^2 + (u_2)^2} \\
&= |t|\sqrt{(u_1)^2 + (u_2)^2} \\
&= |t|\ \|\vec{u}\|
\end{aligned}$$

DEFINITION: The **cross product** of $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is

$$\vec{u} \times \vec{v} = u_1 v_2 - u_2 v_1$$

Note that in 3D, the cross product of two vectors returns a vector rather than a scalar value.

CROSS PRODUCTS AND GEOMETRY

As defined, the cross product finds the area of the parallelogram created by $\vec{u}$ and $\vec{v}$.

EXAMPLE: Let $\vec{u} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$. Find the area of the triangle created by the origin, $\vec{u}$, and $\vec{v}$.

We have that $|\vec{u} \times \vec{v}| = (6)(8) - (0)(0) = 48$. Why absolute value?

This is the area of the parallelogram, so to find the area of the triangle, we need to divide by 2.

$$\frac{|\vec{u} \times \vec{v}|}{2} = \frac{48}{2} = 24$$

EXAMPLE: Let $\vec{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. Find the area of the triangle created by the origin, $\vec{u}$, and $\vec{v}$.

We have that $|\vec{u} \times \vec{v}| = |(-1)(3) - (2)(3)| = |-9| = 9$.

This is the area of the parallelogram, so to find the area of the triangle, we need to divide by 2.

$$\frac{|\vec{u} \times \vec{v}|}{2} = \frac{9}{2} = 4.5$$

The cross product can also be used with the angle the vectors create.

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\ \|\vec{v}\|\ |\sin\theta|$$

Using more math, we can remove the $|\sin\theta|$ to get the result:

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\ \|\vec{v}\|\sqrt{1 - (\hat{u} \cdot \hat{v})^2} \tag{1}$$

EXAMPLE: Let $\vec{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. Find $\|\vec{u} \times \vec{v}\|$ using equation (1).

Start by finding some results: $\|\vec{u}\| = \sqrt{5}$ and $\|\vec{v}\| = \sqrt{18}$.

Therefore, $\hat{u} = \dfrac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\hat{v} = \dfrac{1}{\sqrt{18}} \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

We have $\hat{u} \cdot \hat{v} = \dfrac{1}{\sqrt{5}} \dfrac{1}{\sqrt{18}} [(-1)(3) + (2)(3)] = \dfrac{3}{\sqrt{90}}$.

This means $(\hat{u} \cdot \hat{v})^2 = \left(\dfrac{3}{\sqrt{90}} \right)^2 = \dfrac{9}{90}$.

Putting this all together

$$\begin{aligned} \|\vec{u} \times \vec{v}\| &= \sqrt{5}\sqrt{18}\sqrt{1 - \frac{9}{90}} \\ &= \sqrt{90}\sqrt{\frac{81}{90}} \\ &= \sqrt{90}\frac{\sqrt{81}}{\sqrt{90}} \\ &= \sqrt{81} \\ &= 9 \end{aligned}$$

PROPERTIES OF THE CROSS PRODUCT

Let $\vec{u} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Help discover the following properties of the cross product for yourself.

1. Find $\vec{u} \times \vec{v}$. Find $\vec{v} \times \vec{u}$. What do you notice?

   $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

2. Find $\vec{u} \times \vec{u}$. Find $\vec{v} \times \vec{v}$. What do you notice?

   $\vec{u} \times \vec{u} = 0$

3. Find $(2\vec{u}) \times \vec{u}$. Find $(-5\vec{v}) \times \vec{v}$. What do you notice?

   $t\vec{u} \times \vec{u} = 0, t \in \mathbb{R}$

4. Find $(2\vec{u}) \times \vec{v}$. Find $2(\vec{u} \times \vec{v})$. What do you notice?

   $(t\vec{u}) \times \vec{v} = t(\vec{u} \times \vec{v}), t \in \mathbb{R}$