For the purposes of demonstration last week, our experiment was rushed and not as thorough as it could have been. Therefore, we should take steps to improve on our experimental design and methodology so we can be more confident in our results and better support our conclusions.

Question 1: When we perform an experiment, we want as little error in our measurements as possible. What were the main sources of measurement error in last week's experiment?

1. Drop height relative to catcher's hand likely varied between experimenters.
2. Drop distance was measured to nearest centimetre.
3. Catchers may have been distracted or unprepared.

Question 2: Based on your answers from Question 1, what can we do reduce the measurement error in the reaction time experiment?

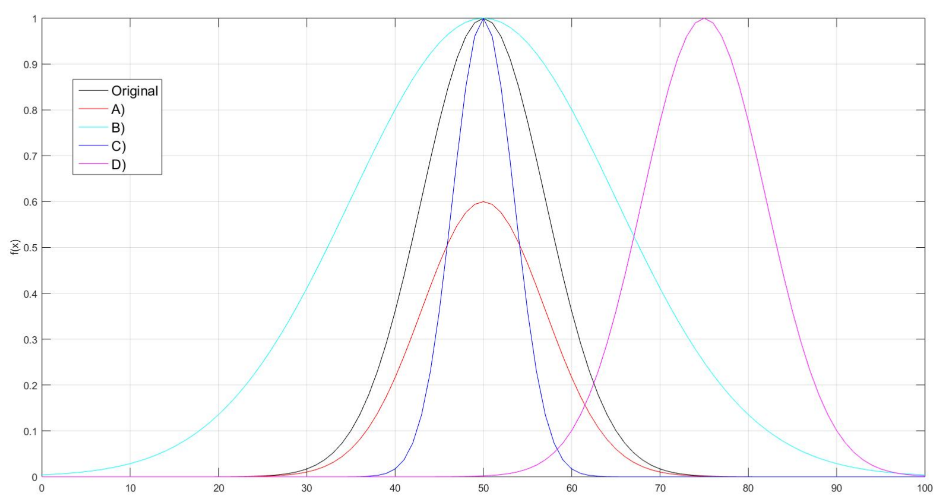
1. Ensure catcher's hand is at the zero mark of ruler, and have catcher and dropper's hands secured to prevent unwanted motion.
2. Measure to the nearest millimetre. The added precision should give a better measure of reaction time, and reduce the random error.
3. Repeat the experiment multiple times to get a better reading of reaction time.

Question 3: What conclusions can you draw from the data if both measurement error and population variation are large?
Answer: We cannot draw any meaningful conclusions since there is too much variation in the data.

Question 4: In the figure below we have a typical Gaussian distribution, which can be expressed by the equation

$$f(x) = Ae^{-\frac{(x-x_0)^2}{2\sigma^2}}, \quad (1)$$

where A is the amplitude, x_0 is the mean, and σ is the standard deviation. Draw the same distribution with A) a smaller amplitude, B) a larger standard deviation, C) a smaller standard deviation, and D) a larger mean.



Question 5: When a ball is dropped from rest at height H, assuming no drag, its vertical position y is given by the equation

$$y = H - gt^2, \tag{2}$$

where g is the acceleration due to gravity and t is the amount of time since the object was dropped. Using a ball, a timer, and some measuring tape, design a experiment to determine the value of g based on this equation. What sources of error will be present in your measurements, and how can you minimize them?

Possible Experiment: In order to determine the value of g, we can drop the ball from several different heights (or just the same height repeatedly), and time how long it takes to reach the floor (where $y = 0$). We can then use the drop height and fall time to solve for g,

$$g = \frac{H}{t^2}$$

By taking a sufficient number of measurements, we should be able to determine an accurate measure of its value and reduce random errors. If we assume that the timer and the measuring tape give accurate enough readings in order to minimize systematic error, our main sources of error will be random. One source of random error will be making sure the timer is stopped as soon as the ball hits the floor. Since this error is associated with how quickly the experimenter can stop the clock and making sure they do so at the right time, we can reduce the relative error by dropping the ball from greater heights. This will ensure that the random error associated with starting and stopping the timer is small compared to the overall fall time. Additionally, if we have a person drop the ball, there is a likely chance the height may vary as they release the object. To minimize this, we can push the ball off of a ledge, as the horizontal motion of the ball will not affect the time it takes to reach the ground.

Probability Distributions

Last time toward the end of the session we measured reaction times and heights. Both are examples of quantities that are easy to assign numbers to. More formally we'd call them *quantifiable*. While some quantities have a correct answer (say your height) others may not have a single right answer. Reaction time is a good example. If we woke you up in the middle of the night and we did the ruler test you'd most likely record a far slower reaction time.
This week we want to talk a little bit about what we do with data once we get it and how mathematics puts a meaning to this process.
The first thing we want to do is assign a probability to something happening. For example we would say the probability of rolling an even number on a single fair six-sided die is 1/2 because $2, 4, 6$ are all even and $1, 3, 5$ are odd (and hence not even). The probability in this case can be worked out as

$$P = \frac{\text{number of all correct possibilities}}{\text{number of all possibilities}}$$

or

$$P = \frac{3}{6}$$

which of course works out to a half.

Problem 1 What is the probability you would roll a 1 on a six-sided die? What about a number less than 3? How do these values change if you are rolling a 20-sided die?

Solution
1 on a six-sided die: 1/6
Number less than 3 on a six-sided die: $2/6 = 1/3$
1 on a 20-sided die: 1/20
Number less than 3 on a 20-sided die: $2/20 = 1/10$

For data we can't quite do this, but we could provide an algorithm. Suppose we asked 100 people their birth month. The number of people born in each month are given in the following table.

J	F	M	A	M	J	J	A	S	O	N	D
10	4	7	8	8	12	9	11	8	9	8	6

We can then define the probability of being born in a given month as the number measured divided by the total (100). So for January we would get 0.1 while for December we would get 0.06.

Problem 2 Work out the probabilities for each month. Would you expect them all to be the same? What might make them slightly different? How close are the measured probabilities to what you would expect?

Solution

J	F	M	A	M	J
$10/100 = 0.1$	$4/100 = 0.04$	$7/100 = 0.07$	$8/100 = 0.08$	$8/100 = 0.08$	$12/100 = 0.12$

J	A	S	O	N	D
$9/100 = 0.09$	$11/100 = 0.11$	$8/100 = 0.08$	$9/100 = 0.09$	$8/100 = 0.08$	$6/100 = 0.06$

Assuming that births occur randomly throughout the year, we'd expect the probability of being born in a given month to be about $1/12 = 0.08\bar{3}$. Of course since the months have different numbers of days, the probabilities would be $31/365 = 0.089$ for months with 31 days, 0.0822 for months with 30 days, and $28/365 = 0.0767$ for February. These numbers will be slightly different for leap years. However, birthrates are not purely random, and the number of people born at different times during the year will depend on a number of factors.

If you've worked out Problem 2 you'll find that, in fact, you would expect most months to be basically the same. However, they are clearly not in the sample data. What does this mean? Well it means that some of the attractive parts of mathematics need to be loosened a bit when we deal with the real world. Or put another way, maybe to get some of the pretty mathematics back we need a way to tame the wildness of real world data. One thing we did last week was to take the measurement several times. This would help account for some of the error in the experiment. To see why let's consider a sample set of reaction times measured in centimeters. The first time we measure 12 cm, the second time we measure 8 cm, and the full set of ten measurements is given by

$$(12\,\text{cm}, 8\,\text{cm}, 11\,\text{cm}, 13\,\text{cm}, 11\,\text{cm}, 11\,\text{cm}, 6\,\text{cm}, 20\,\text{cm}, 10\,\text{cm}, 14\,\text{cm}).$$

We can compute the mean of the measurements to find it is 11.6 cm. So our first measurement of 12 cm is kind of lucky in how close it is. Are there any other observations we can make? Well, there are two obvious outliers, namely 6 cm and 20 cm. These might be because the subject got twitchy and started closing their fingers before the ruler got dropped, then got reprimanded for cheating and so was slow the next time around.
Let's try to do a bit more math with the data. As it stands, there is not much more we can say, but what if we only cared about reactions in a certain range. Let's divide the reaction times into bins, $(1,2,3)$, $(4,5,6)$, $(7,8,9)$, $(10,11,12)$, $(13,14,15)$, $(16,17,18)$, $(19,20,21)$. The number of measurements in each of these bins is

$$(0,1,1,5,2,0,1)$$

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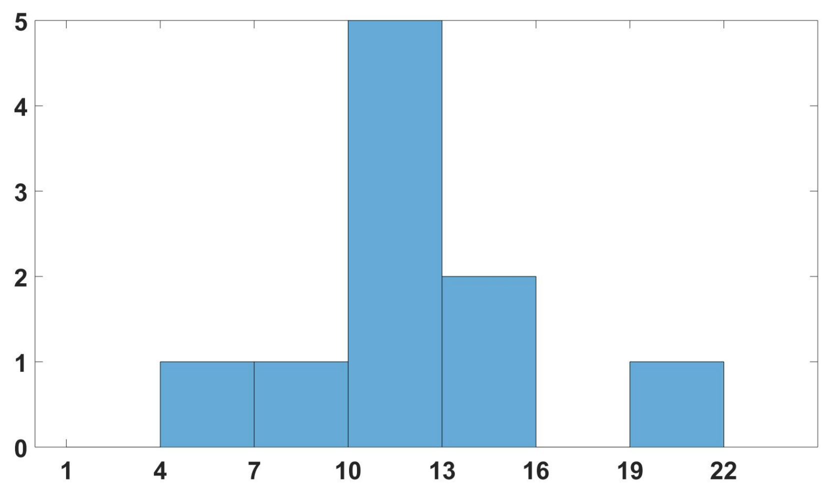
Problem 3 Make a bar graph of the bin counts. Is there anything special about the graph?

Solution



In these graphs, we arrange the bars so that they start at the first number of each corresponding bin. The bin with the greatest count contains the mean.

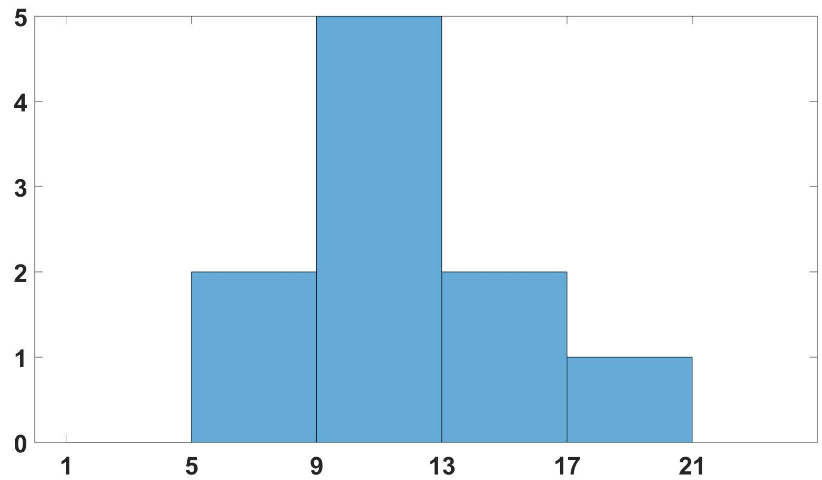
You should have noticed that one of the bins is quite exceptional since it has a much larger count than the others. To create a probability simply divide the bin count by the number of measurements and find

$$(0, 0.1, 0.1, 0.5, 0.2, 0, 0.1).$$

You can confirm for yourself that the probability adds up to 1.

Problem 4 Repeat the exercise but use bins of width 4. How different do the bar graphs look?

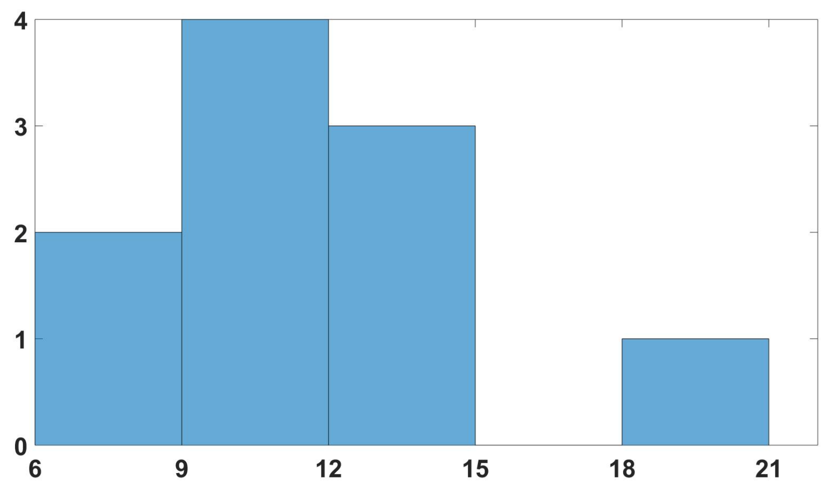
Solution



The largest bar corresponds to the $(9, 10, 11, 12)$ bin, which again contains the mean. There are no gaps between bars in this case, and the graph is slightly more symmetric.

Problem 5 Repeat the exercise using bins of width 3 again but this time start them at the lowest value so that the first bin is $(6, 7, 8)$. How does this compare to the previous bins of width 3 case?

Solution

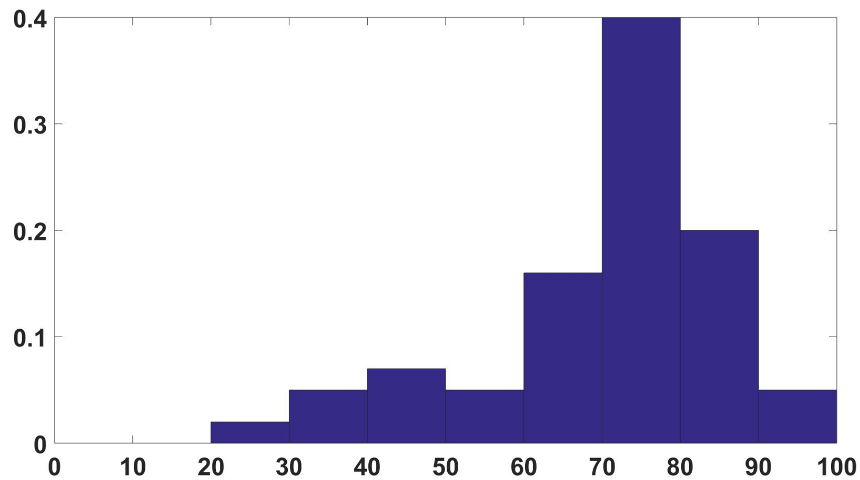


The largest bin has a lower count than in the previous bin of width 3 graph, but the adjacent bars are larger. This graph also has a gap in the data due to the 20 cm measurement.

OK, now we have the ammunition to actually ask some real math questions. Say we have test results for a standardized test marked out of 100 (integer grades only). We have a thousand students tested so we trust our results. We bin by 10s (i.e., $0 - 9\%$, $10 - 19\%$, etc.), so we have grades in the sixties, seventies and so on. Our resulting probabilities are $(0, 0, 0.02, 0.05, 0.07, 0.05, 0.16, 0.4, 0.2, 0.05)$.

Problem 6 Sketch the bar graph of probabilities. Does it have a particular shape?

Solution



Now let's say we want to know the probability of a failed grade (below 50). How do we find it? Well, we assumed that the bins were all the same size (very important) so we could just add all the probabilities and find 0.14 or a 14% failure rate.

Problem 7 Find the probability of i) grades in the 80s or 90s, ii) grades in the 60s or 70s.

Solution

i) $P_{80\,\mathrm{or}\,90} = P_{80} + P_{90} = 0.2 + 0.05 = 0.25$
ii) $P_{60\,\mathrm{or}\,70} = P_{70} + P_{80} = 0.16 + 0.4 = 0.56$

We mentioned above that we used the fact that the bins were equal sized. Let's say we now created a data set in which we bin the failures separately so that we have probabilities

$$(0.14, 0.05, 0.16, 0.4, 0.2, 0.05).$$

The problem is that writing it like this does not remind us of how big the first bin is.

Problem 8 By appropriately accounting for the bin size, calculate the correct probability of a score that is either failing or in the 90s. Explain your work.

Solution $P_{\mathrm{Fail\,or}\,90} = P_{\mathrm{Fail}} + P_{90} = 0.14 + 0.05 = 0.19$
Because the probability of a failure is already a sum of all the probabilities of a mark below 50, we can add it directly to the probability of a 90 in order to determine the probability of a score that is either failing or in the 90s.