Intermediate Math Circles
Wednesday March 09, 2016
Introduction to Vectors III

Review

If $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, then we have the following properties:

1. Vector Addition: $\vec{u} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$

2. Scalar Multiplication: $t\vec{v} = t\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} tv_1 \\ tv_2 \end{bmatrix}, t \in \mathbb{R}$

3. Norm/Length of a Vector: $\|\vec{v}\| = \sqrt{(v_1)^2 + (v_2)^2}$

4. Unit Vector: $\hat{v} = \frac{1}{\|\vec{v}\|}\vec{v}$

5. Dot Product: $\vec{u} \bullet \vec{v} = u_1v_1 + u_2v_2$

6. Cross Product: $\vec{u} \times \vec{v} = u_1v_2 - u_2v_1$

Example Let $\vec{u} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, find $\|\vec{u} \times \vec{v}\|$ using the formula

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \; \|\vec{v}\|\sqrt{1 - (\hat{u} \bullet \hat{v})^2}$$

Solution
Start by finding some results:

$$\|\vec{u}\| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

and

$$\|\vec{v}\| = \sqrt{3^2 + 4^2} = 5$$

Therefore, $\hat{u} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and $\hat{v} = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

$$\begin{aligned} \hat{u} \bullet \hat{v} &= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{5}\right)[(-1)(3) + (-1)(4)] \\ &= \frac{1}{5\sqrt{2}}(-7) \\ &= -\frac{7}{\sqrt{(25)(2)}} \\ &= -\frac{7}{\sqrt{50}} \end{aligned}$$

Therefore,

$$\begin{aligned}(\hat{u} \bullet \hat{v})^2 &= \left(\frac{-7}{\sqrt{50}}\right)^2 \\ &= \frac{49}{50}\end{aligned}$$

$$\begin{aligned}\|\vec{u} \times \vec{v}\| &= \|\vec{u}\|\ \|\vec{v}\|\sqrt{1-\left(\hat{u} \bullet \hat{v}\right)^2} \\ &= 5\sqrt{2}\sqrt{1-\frac{49}{50}} \\ &= \sqrt{50}\sqrt{\frac{1}{50}} \\ &= \sqrt{50}\frac{\sqrt{1}}{\sqrt{50}} \\ &= 1\end{aligned}$$

Notice from the cross product formula

$$\vec{u} \times \vec{v} = (-1)(4) - (-1)(3) = -1$$

Example Prove $\vec{u} \bullet \vec{u} = 0$ only if $\vec{u} = \vec{0}$.

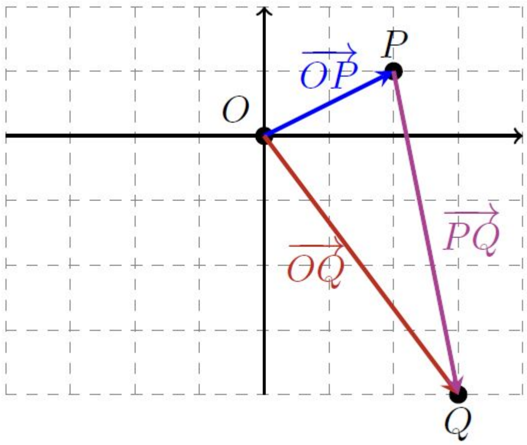
Solution

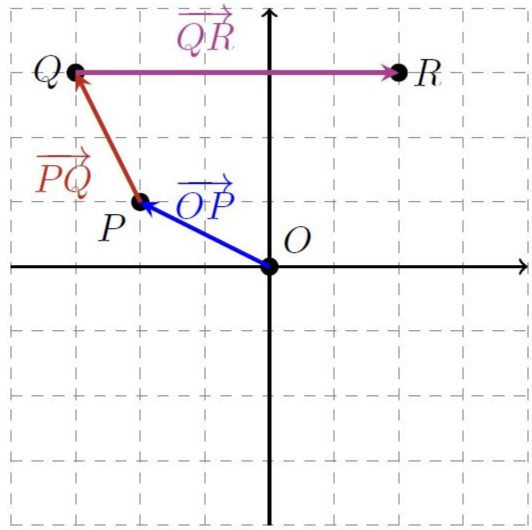
Suppose $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$. Using the definition of the dot product

$$\begin{aligned}\vec{u} \bullet \vec{u} &= 0 \\ (a)(a) + (b)(b) &= 0 \\ a^2 + b^2 &= 0 \\ a^2 &= -b^2\end{aligned}$$

But a^2 and b^2 are always positive.
The only way both sides can be equal ($LHS = RHS$) is if $a = b = 0$. That is, $\vec{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$.

Tip to Tail



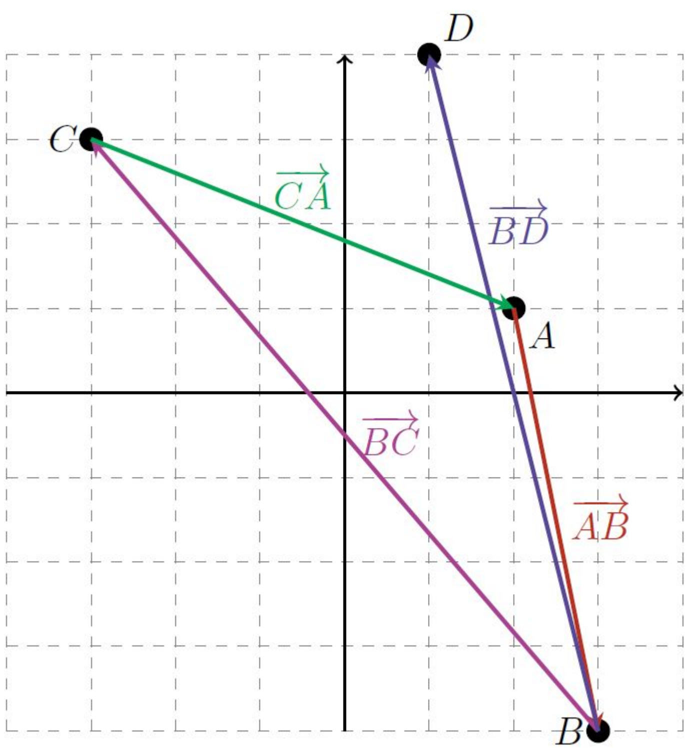


Definition The **directed line segment** from a point P to a point Q is drawn as an arrow starting at point P with tip at point Q. It is denoted $\overrightarrow{PQ}$**.**

Example On the second graph, plot $\overrightarrow{OP}$, $\overrightarrow{PQ}$, and $\overrightarrow{QR}$ if $P = (-2, 1)$, $Q = (-3, 3)$, and $R = (2, 3)$.

Definition Two directed line segments, $\overrightarrow{PQ}$ and $\overrightarrow{RS}$, are **equivalent** if $\vec{q} - \vec{p} = \vec{s} - \vec{r}$ in which case we will write $\overrightarrow{PQ} = \overrightarrow{RS}$

Example Plot and label the vectors $\vec{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$, and $\vec{d} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.



Find and plot

- $\overrightarrow{AB}$
- $\overrightarrow{BC}$
- $\overrightarrow{BD}$
- $\overrightarrow{CA}$

Extension into 3D

We define $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $t \in \mathbb{R}$. Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

- $\vec{u} + \vec{v}$

  $= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$

- $\vec{u} + \vec{v}$

  $= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1+4 \\ 2+5 \\ 3+6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$

- $t\vec{u}$

  $= t\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} tu_1 \\ tu_2 \\ tu_3 \end{bmatrix}, t \in \mathbb{R}$

- $-\frac{1}{2}\vec{u}$

  $= -\frac{1}{2}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1 \\ -3/2 \end{bmatrix}$

- $\vec{u} \bullet \vec{v}$

  $= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \bullet \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = u_1v_1 + u_2v_2 + u_3v_3$

- $\vec{u} \bullet \vec{v}$

  $= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

  $= (1)(4) + (2)(5) + (3)(6) = 32$

- $\|\vec{u}\|$

  $= \sqrt{\vec{u} \bullet \vec{u}} = \sqrt{(u_1)^2 + (u_2)^2 + (u_3)^2}$

- $\|\vec{u}\|$

  $= \sqrt{1^2 + 2^2 + 3^3} = \sqrt{1 + 4 + 9} = \sqrt{13}$

- $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$

  $= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}$

- $d(\vec{u}, \vec{v})$

  $= \sqrt{(1-4)^2 + (2-5)^2 + (3-6)^2}$

  $= \sqrt{9+9+9} = \sqrt{27} = 3\sqrt{3}$

- $\hat{u} = \frac{1}{\|\vec{u}\|}\vec{u}$

- $\hat{u} = \frac{1}{\sqrt{13}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{13} \\ 2/\sqrt{13} \\ 3/\sqrt{13} \end{bmatrix}$

- $\vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$

  ⋆ Note this is different from the 2D case.

- $\vec{u} \times \vec{v} = \begin{bmatrix} (2)(6) - (3)(5) \\ (3)(4) - (1)(6) \\ (1)(5) - (2)(4) \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$

  The cross product gives a vector result rather than a scalar!

- $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \; \|\vec{v}\| \sqrt{1 - (\hat{u} \bullet \hat{v})^2}$

  or

  $\|\vec{u} \times \vec{v}\|^2 = (\vec{u} \bullet \vec{u})(\vec{v} \bullet \vec{v}) - (\vec{u} \bullet \vec{v})^2$

- $\|\vec{u} \times \vec{v}\| = \sqrt{\begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \bullet \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}}$

  $= \sqrt{9+36+9} = \sqrt{54} = 3\sqrt{6}$