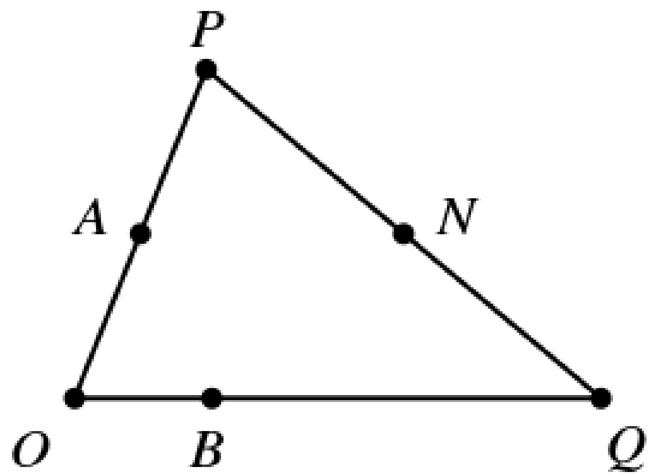
# Intermediate Math Circles
# Wednesday March 09, 2016
# Introduction to Vectors III

1. In the diagram, $A$ is the midpoint of $OP$ and $N$ is the midpoint of $PQ$, with $\overrightarrow{OB} = \frac{1}{4}\overrightarrow{OQ}$. Let $\vec{u} = \overrightarrow{OA}$ and $\vec{v} = \overrightarrow{OB}$. Write the following vectors in terms of $\vec{u}$ and $\vec{v}$.




(a) $\overrightarrow{AP}$

$\vec{u}$

(b) $\overrightarrow{QO}$

$-4\vec{v}$

(c) $\overrightarrow{PB}$

$-2\vec{u} + \vec{v}$

(d) $\overrightarrow{QP}$

$2\vec{u} - 4\vec{v}$

(e) $\overrightarrow{NQ}$

$-\vec{u} + 2\vec{v}$

(f) $\overrightarrow{ON}$

$\vec{u} + 2\vec{v}$

2. What value of $a$ makes $\begin{bmatrix} -8 \\ a \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ a \end{bmatrix}$ orthogonal?

We use the dot product:

$$\begin{aligned} 0 &= \begin{bmatrix} -8 \\ a \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ -1 \\ a \end{bmatrix} \\ 0 &= (-8)(2) + (a)(-1) + 3(a) \\ 0 &= -16 - a + 3a \\ 16 &= 2a \\ 8 &= a \end{aligned}$$

3. Let $\vec{u} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$. If $\vec{z} = \vec{u} \times \vec{v} = \begin{bmatrix} 2 \\ 16 \\ 11 \end{bmatrix}$, what is $\vec{u} \bullet \vec{z}$? What is $\vec{v} \bullet \vec{z}$? What is special about the vector created by the cross product of two vectors?

$$\vec{u} \bullet \vec{z} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 16 \\ 11 \end{bmatrix} = (2)(2) + (-3)(16) + (4)(11) = 4 - 48 + 44 = 0$$

$$\vec{v} \bullet \vec{z} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 16 \\ 11 \end{bmatrix} = (3)(2) + (1)(16) + (-2)(11) = 6 + 16 - 22 = 0$$

The cross product is orthogonal to the two vectors. Without proof, this is true in general.

4. Consider the points $P(1, 5, 7)$, $Q(2, 4, 3)$, and $R(3, 3, -1)$. What is the relationship between $\overrightarrow{PQ}$ and $\overrightarrow{RQ}$?

   We calculate the direction vectors:

$$\overrightarrow{PQ} = \vec{q} - \vec{p} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}$$

$$\overrightarrow{RQ} = \vec{q} - \vec{r} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$

   Therefore, $\overrightarrow{PQ} = -\overrightarrow{RQ}$.

5. Find the vector equation for the line that passes through $P(-5, 2, 10)$ and $Q(3, -4, -4)$.

   We calculate the direction vector:

$$\overrightarrow{PQ} = \vec{q} - \vec{p} = \begin{bmatrix} 3 \\ -4 \\ -4 \end{bmatrix} - \begin{bmatrix} -5 \\ 2 \\ 10 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \\ -14 \end{bmatrix}$$

   Therefore, a vector equation for the line is

$$\ell = \begin{bmatrix} -5 \\ 2 \\ 10 \end{bmatrix} + t \begin{bmatrix} 8 \\ -6 \\ -14 \end{bmatrix}, t \in \mathbb{R}$$

   Note that we can use the point $Q$ instead of point $P$ or, in fact, any other point on the line as the position vector.

6. Find the distance between the points $P(2, 3, 0)$ and $Q(-1, 2, 4)$.

   We have $\overrightarrow{PQ} = \vec{q} - \vec{p} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix}$.

$$\|\overrightarrow{PQ}\| = \sqrt{\overrightarrow{PQ} \bullet \overrightarrow{PQ}} = \sqrt{(-3)^2 + (-1)^2 + 4^2} = \sqrt{26} \approx 5.1$$

7. Let $\vec{u} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. Find the following:

(a) $\vec{u} + \vec{v}$

$$\vec{u} + \vec{v} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

(b) $\frac{3}{2}\vec{u}$

$$\frac{3}{2}\vec{u} = \frac{3}{2}\begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3/2 \\ -3 \end{bmatrix}$$

(c) $\vec{u} \bullet \vec{v}$

$$\vec{u} \bullet \vec{v} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = (4)(0) + (1)(1) + (-2)(2) = 1 - 4 = -3$$

(d) $d(\vec{u}, \vec{v})$

$$(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(4-0)^2 + (1-1)^2 + (-2-2)^2} = \sqrt{16 + 16} = \sqrt{32} = 4\sqrt{2}$$

(e) $\|\vec{u}\| + \|\vec{v}\|$

$$\begin{aligned} \|\vec{u}\| + \|\vec{v}\| &= \left\| \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\| \\ &= \sqrt{4^2 + 1^2 + (-2)^2} + \sqrt{0^2 + 1^2 + 2^2} \\ &= \sqrt{21} + \sqrt{5} \end{aligned}$$

(f) $\hat{u}$ and $\hat{v}$

$$\hat{u} = \frac{1}{\sqrt{21}}\begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} \qquad \hat{v} = \frac{1}{\sqrt{5}}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

(g) $\vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} = \begin{bmatrix} (1)(2) - (-2)(1) \\ (-2)(0) - (4)(2) \\ (4)(1) - (1)(0) \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ 4 \end{bmatrix}$$

(h) $\|\vec{u} \times \vec{v}\|$

$$\|\vec{u} \times \vec{v}\| = \sqrt{4^2 + (-8)^2 + 4^2} = \sqrt{96} = 4\sqrt{6}$$

8. (Harder) Let $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$.

   (a) Find $\vec{u} \bullet (\vec{v} \times \vec{w})$. This is the volume of the 3D parallelogram (known as a parallelepiped) created by the three vectors.

$$\vec{v} \times \vec{w} = \begin{bmatrix} (-2)(4) - (-1)(1) \\ (-1)(2) - 3(4) \\ (3)(1) - (-2)(2) \end{bmatrix} = \begin{bmatrix} -7 \\ -14 \\ 7 \end{bmatrix}$$

   Therefore,

$$\vec{u} \bullet (\vec{v} \times \vec{w}) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} -7 \\ -14 \\ 7 \end{bmatrix} = 14$$

   (b) Find $\|\vec{v} \times \vec{w}\|$ given the formula

$$\|\vec{v} \times \vec{w}\|^2 = (\vec{u} \bullet \vec{u})(\vec{v} \bullet \vec{v}) - (\vec{u} \bullet \vec{v})^2$$

   Calculating the dot products, we have

$$\vec{v} \bullet \vec{v} = 14 \qquad \vec{w} \bullet \vec{w} = 21 \qquad \vec{v} \bullet \vec{w} = 0$$

   Therefore,

$$\|\vec{v} \times \vec{w}\|^2 = (14)(21) - (0)^2 = 294$$

**Sudoku to end the day**

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|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 7 | 5 | 8 | 9 | 6 | 2 | 4 |
| 4 | 8 | 6 | 7 | 3 | 2 | 1 | 9 | 5 |
| 5 | 9 | 2 | 1 | 4 | 6 | 3 | 8 | 7 |
| 9 | 2 | 8 | 4 | 6 | 7 | 5 | 3 | 1 |
| 6 | 4 | 5 | 3 | 9 | 1 | 2 | 7 | 8 |
| 3 | 7 | 1 | 2 | 5 | 8 | 9 | 4 | 6 |
| 2 | 5 | 4 | 8 | 1 | 3 | 7 | 6 | 9 |
| 8 | 6 | 3 | 9 | 7 | 5 | 4 | 1 | 2 |
| 7 | 1 | 9 | 6 | 2 | 4 | 8 | 5 | 3 |