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Grade 7/8 Math Circles

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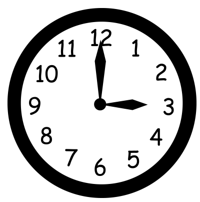
Modular Arithmetic

1 Introduction: The 12-hour Clock



Question: If it's 7 pm now, what time will it be in 7 hours?
How about in 30 hours?
Now, this is a simple example that we're all familiar with, but how did we actually calculate this? Can you simplify this into 2 or 3 simple steps?

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2 Review of Divisibility

Definition: An integer x is divisible by an integer n if $x \div n$ is an integer *(ie. there is no remainder when x is divided by n)*. We write $x \mid n$, which is read "x is divisible by n".
Can you think of another way to ask "Is x divisible by n?"

Exercise 1

(a) Is 15 divisible by 3?

(b) Is 75 divisible by 2?
Note: this is another way of asking if 75 is even

(c) Is 50 divisible by 15?

(d) is 150 divisible by 3?

3 Modular Operator

The modular operator might seem a little intimidating at first, but it's really not. All it does is, given 2 integers (x and n), it produces the remainder when the first number is divided by the second.
Notation: $x \pmod{n} = r$
This means that when x is divided by n, there is a remainder of r.
We say: "x modulo n is equal to r".

Examples:

(a) $7 \pmod{4} =$

(b) $15 \pmod{3} =$

(c) $19 \pmod{4} =$

(d) $21 \pmod{5} =$

Exercise 2: Calculate each of the following.

(a) $7 \pmod{5} =$

(b) $8 \pmod{4} =$

(c) $8 \pmod{3} =$

(d) $17 \pmod{8} =$

(e) $37 \pmod{6} =$

(f) $124 \pmod{60} =$

3.1 Modular Addition

Modular addition is actually quite straight forward.
For example:

$$(1+2) \pmod{4} = 3$$
$$(4+5) \pmod{5} = 9 \pmod{5} = 4$$

Pretty simple right? What if it got more complicated though, like this one?

$$(187468 + 847361) \pmod{2} = \underline{\hspace{2cm}}$$

This is not an easy calculation, unless you have a calculator, but that defeats the purpose of modular arithmetic, which is to simplify complicated calculations.
So, we propose an idea: What if we were to calculate each number with respect to that modulo before we add them together? Now this complicated question becomes really simple:

$$(187468 + 847361) \pmod{2} = (0 + 1) \pmod{2} = 1$$

The next natural thought would be to define modular addition as the following:

$$(x+y) \pmod n = x \pmod n + y \pmod n$$

For example: $(7+6) \pmod 5 = 7 \pmod 5 + 6 \pmod 5 = (2+1) \pmod 5 = 3$
Try this one: (19+28) $\pmod 5 =$ ________________________________
As we see with this example, we can't just calculate each number with respect to $\pmod n$ and add them together, sometimes we are required to simplify the sum in with respect to $\pmod n$ after we sum them. So we define modular addition as:

$$\boxed{(x+y) \pmod n = [x \pmod n + y \pmod n] \pmod n}$$

In general, we know we've simplified it as much as possible when the result is ____________

Exercise 3: Calculate each of the following.

(a) $5+9 \pmod 8 =$

(b) $43+37 \pmod{10} =$

(c) $124+199 \pmod 5 =$

(d) $34+121 \pmod{11} =$

3.2 Modular Multiplication

Modular multiplication is very similar to modular addition. We define it as:

$$\boxed{(x \times y) \pmod n = [x \pmod n \times y \pmod n] \pmod n}$$

Exercise 4: Calculate each of the following.

(a) $5 \times 9 \pmod 8 =$

(b) $7 \times 15 \pmod 7 =$

(c) $5782 \times 2579 \pmod{10} =$

(d) $603 \times 123 \pmod{60} =$

(e) $16 \times 25 \pmod{12} =$

(f) $34 \times 122 \pmod{11} =$

4 Common Bases

Modular artihmetic are used in the real world on a daily basis. As we saw in the introduction of this lesson, base 12 is a common one used in analog clocks. Here are some other commonly used bases:

Base	**Application**	**Example**
2	Even/odd numbers Binary codes	A number n is even if $n \pmod 2 = 0$, and odd otherwise. We also use base 2 when using converting from binary to decimal form, as we will see later.
4	Years between 2 consecutive leap years (in general)	If any given year n is either $[n \pmod{400} = 0]$ or $[n \pmod 4 = 0$ and $n \pmod{100} \neq 0]$ then it is a leap year, otherwise it isn't.
7	Days in a week	If today is Sunday, then in 16 days it will be a Tuesday (since $16 \pmod 7 = 2$).
10	Metric measurements	We use base 10 when converting between metric measurments, such as metres to millimetres.
12	Hours on an analog clock	If it's 7 pm now, it will be 2 am in 7 hours (since $(7+7) \pmod{12} = 14 \pmod{12} = 2$).
24	Hours in a day	If its 2 pm now, in 54 hours it will be 8 am (since $(54+2) \pmod{24} = 8$).
28, 29, 30, 31	Days in a month	If today is the 4^{th} of April, then it will be the 8^{th} of May in 34 days.
52	Weeks in a year	If today is the 6^{th} week of the year, then it will be the 16^{th} week of next year in 62 weeks.
60	Seconds in a minute and minutes in an hour	155 seconds is equivalent to 2 minutes and 35 seconds.
100	Years in a century	In 344 years, it will be the 60^{th} year of that century, since $(344+2016) \pmod{100} = 60$.
360	Degrees in a full circle	Rotating 420° is equivalent to rotating 60° since $420 \pmod{360} = 60$.
365	Days in a year	If today is the 65^{th} day of the year, then in 750 days, it will be 85^{th} day of that year.

5 Binary Numbers and Codes

A binary code is any system that only uses 2 states: 1/0, on/off, true/false etc.
A binary number is any number containing only 1's and 0's. These are all examples of binary numbers:
101 000000 1111111 10001001010010 10001111101010 0101010101010
Binary numbers have all sorts of applications, many of which are used on a daily basis, like:

- Computers
- Calculators
- TV's
- Barcodes
- CD's and DVD's
- Braille

Binary codes are also used in many work fields, such as computer science, software engineering and electrical engineering - and basically all other fields of engineering too!
There are multiple ways to express a binary code, the two most common forms of writing a binary code using numbers are 'Decimal form' and 'Binary form'.
For example: 1101 in binary form becomes 13 in decimal form. And 1001 becomes 9.
Now, the conversion between these may not be obvious, but it's pretty easy. Before we jump into converting between binary and decimal forms, let's do a quick review on exponents:

$$x^0 = 1$$
$$x^1 = x$$
$$x^2 = x \times x$$
$$x^3 = x \times x \times x$$
$$x^4 = x \times x \times x \times x$$
$$x^5 = x \times x \times x \times x \times x$$

and so on... (for any x)

Also, fill out this table, it will be very useful for the rest of the lesson.

$\boldsymbol{n}$	0	1	2	3	4	5	6	7	8
$\boldsymbol{2^n}$	$2^0 =$	$2^1 =$	$2^2 =$	$2^3 =$	$2^4 =$	$2^5 =$	$2^6 =$	$2^7 =$	$2^8 =$

5.1 Converting Binary to Decimal

To convert a binary number to its decimal form, follow these 3 simple steps:

① Write out the number - but leave lots of space between your digits, like this:

$$1 \qquad\qquad 0 \qquad\qquad 0 \qquad\qquad 1$$

② Multiply each number by a 2, and starting with an exponent of 0 on the very last 2, and increase the exponent by 1 each time, like this:

$$[1 \times (2^3)] \quad [0 \times (2^2)] \quad [0 \times (2^1)] \quad [1 \times (2^0)]$$

③ Sum them up and calculate:

$$[1 \times (2^3)] + [0 \times (2^2)] + [0 \times (2^1)] + [1 \times (2^0)]$$

$$=[1 \times (8)] + [0 \times (4)] + [0 \times (2)] + [1 \times (1)]$$

$$=8 + 0 + 0 + 1 = 9$$

Exercise 5: Convert each of the follwing binary numbers to decimal form.

(a) 110 $\rightarrow$

(b) 101 $\rightarrow$

(c) 00111 $\rightarrow$

(d) 100001 $\rightarrow$

5.2 Converting Decimal to Binary

Now this is the part where modular arithmetic comes in handy!
We know that if we compute any number (mod 2) it will either be 0 or 1, and so that's exactly what we use for converting decimal numbers to binary.
Basically, we compute our number (mod 2) and that will be our last digit. Then we compute our quotient (mod 2) and place that as our 2^{nd} last digit, and so on until our quotient is 0.
For example: Converting 13 to binary form, we would do the following.

$$\begin{aligned} 13 &= 2(6) + 1 \quad \Rightarrow \quad 13 \pmod 2 = 1 \\ 6 &= 2(3) + 0 \quad \Rightarrow \quad 6 \pmod 2 = 0 \\ 3 &= 2(1) + 1 \quad \Rightarrow \quad 3 \pmod 2 = 1 \\ 1 &= 2(0) + 1 \quad \Rightarrow \quad 1 \pmod 2 = 1 \end{aligned}$$

Now reading from the bottom up, 13 in decimal form is 1101 in binary form.
Note: Your last step should ALWAYS be the same as the one above.
Exercise 6: Convert each of the following numbers to binary form.

(a) 76 (b) 193 (c) 97 (d) 255

6 Problem Set

1. Calculate each of the following

   (a) $100 \pmod 3 =$

   (b) $451 \pmod 5 =$

   (c) $490 \pmod 7 =$

   (d) $234 \pmod 4 =$

   (e) $478 \pmod 6 =$

   (f) $582 \pmod 9 =$

   (g) $679 \pmod 8 =$

   (h) $12 + 18 \pmod 9 =$

   (i) $73 + 58 \pmod{12} =$

   (j) $74 \times 93 \pmod{13} =$

   (k) $33 \times 266 \pmod{26} =$

2. Complete the following table by either converting the given binary number to decimal form or vice versa.

	Binary	Decimal
(a)	0101	
(b)	100111	
(c)		15
(d)		20
(e)	1010101	
(f)	10010110	
(g)		45
(h)		59

3. If my birthday was on Monday, January 5, 2015, what day of the week will my birthday be on this year (2016)?

4. If Mary's birthday was on a Thursday in 2014, what day of the week will her birthday be on next year (2017)?

5. Using a regular deck of 52 cards, I dealt all the cards in the deck to 3 people (including myself). Were the cards dealt evenly?

6. A litre of milk is 4 cups, and one cake recipe uses 3 cups. If I have 8 litres of milk, how many cakes can I make? And how many cups of milk will be leftover, if any?

7. I bought as many mini-erasers as possible at 25 cents each and spent the rest of my money on paperclips at 3 cents each. How many of each did I buy given that I have $1.70? Is there anything leftover? *(Assume there's no tax.)*

8. I have 5 trays with 6 muffins each that I divided evenly among 4 of my friends, and I ate the leftovers. How many muffins did each of my friends eat? How many muffins did I eat?

9. If Math Circles started on Tuesday, February 2^{nd}, 2016, and lasts for 44 days, what day will it end? *(Give the full date.)*
   Note that 44 us NOT *the number of classes there are, rather it is the number of days in between the first and last day of Math Circles.*

10. If I celebrated my 16^{th} birthday on Wednesday, February 10^{th}, 2016, what day of the week was I born? *(Don't forget about leap years!)*

11. Mary is facing South and rotates 2295° clockwise. Which direction is she facing now?

12. (a) How many different 5-digit binary numbers are there?
    (b) How many different 5-digit binary numbers are there that have 1 as the last digit?

13. Look back at the table titled "Common Bases".
    (a) Was the year 1900 a leap year?
    (b) Was the year 2000 a leap year?
    (c) Is the year 2100 going to be a leap year?
    (d) Is the year 2200 going to be a leap year?

14. There are seven stacks of coins that look the same. Each stack has exactly 100 coins. There are two stacks that have counterfeit coins, and all 100 coins in each of those two stacks are counterfeit. Your task is to figure out which two of the seven stacks contain the counterfeit coins.
    The counterfeit coins weigh 11g each, while the real coins weigh 10g each. You have an electric balance, but you can only use it to make one measurement.
    How can you determine which two stacks contain the counterfeit coins with only one use of the balance? Explain why the strategy works. *(Hint: Think about taking different numbers of coins from each of the stacks and placing them on the balance together. Think about the important numbers in the binary number system.)*