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Grade 7/8 Math Circles

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Mathematical Thinking

Today we will take a look at some of the people that laid the foundations of mathematics and discuss some interesting proofs and problems that demonstrate how to think mathematically. As we will learn, thinking mathematically generally involves applying logical reasoning to answer a question.

Euclid (4th – 3rd century BCE)

Often called the "father of geometry", Euclid's *Elements* covered many areas of math (including number theory and algebra) and was one of the primary sources of mathematical knowledge from its publication in 300 BCE until the early 20th century. One of his many achievements includes describing **Euclidean geometry**, which was for many centuries considered to be the only geometry possible and the one that you are most familiar with.

Another one of his major achievements is his proof of the **Infinitude of Primes** (below). A **prime number** is any number greater than one who's only positive divisors are one and itself. 2 and 13 are a prime numbers. 6 is not a prime number because it can be divided evenly by 2.

Theorem: There are infinitely many primes.

Proof: Suppose the opposite, that is, that there are a finite number of prime numbers. Call them $p_1, p_2, p_3, p_4, \ldots, p_n$ (all primes that exist). Then the number P below is *not prime* because it is greater than all primes that exist:

$$P = (p_1 \times p_2 \times p_3 \times \ldots \times p_n) + 1$$

Every prime number, when divided into this number, leaves a remainder of one. Therefore, it has no prime factors (no prime number can be divided into it and leave a remainder of zero). This is impossible because the **Fundamental Theorem of Arithmetic** implies that every integer greater than one has at least one prime factor.

Therefore, our assumption that there are a finite number of primes is false and there must be infinitely many primes.

Note: Every prime number has exactly one prime factor: itself.

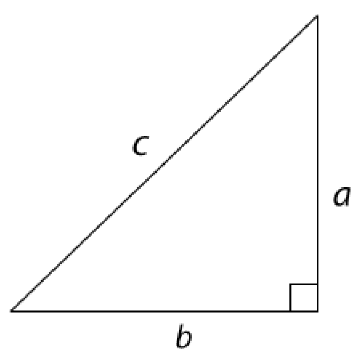
Known as **Euclid's Theorem**, it is a fundamental statement in number theory. The *Elements* is still considered to be a breakthrough in its application of logic to mathematics and has been enormously influential in math and science for its axiomatic structure (logical reasoning).

Pythagoras (570 – 495 BCE)

The first man to call himself a philosopher ("lover of wisdom"), Pythagoras was a thinker ahead of his time. He settled in Crotona, a Greek city in Italy, where he gathered a group of wealthy men and founded the mysterious Pythagorean Brotherhood. They studied arithmetic, music, geometry, and astronomy. Their astronomy had advanced to the point of considering that the earth was a sphere. They believed what we know today, that nature and the whole of reality has an underlying mathematical structure and devoted themselves to revealing it. The brotherhood was so politically and mathematically influential that they were forced to disperse when a prominent citizen got angry that he was denied membership. The brotherhood was heavily persecuted and Pythagoras eventually died in exile.

Some of Pythagoras' achievements include showing that the interior angles of a triangle always added up to 180 degrees and proving the Pythagorean theorem (which you will also do in the activities). The **Pythagorean Theorem** states that for any right angle triangle with side-lengths a, b, and c where c is the **hypoteneuse** (the side opposite to the right angle), it is always true that

$$a^2 + b^2 = c^2$$



The Birthday Paradox

Consider this: an old, ragged traveller gathers you and 22 strangers together in a room. None of you know how you got there or who the traveller is. He tells you all that there is a greater than 50% chance that two of you share a birthday and then disappears in a puff of smoke. Bewildered, you and the people in the room start discussing this strange man and his strange idea. Surely he's joking, right? There are 365 days in a year and the probability that two people in this very small group share a birthday must be very small. As a mathematician, you decide to derive a proof to see if this man's statement is true.

Let's figure out how many people you have to get in a room before the chance of them sharing a birthday is greater than 50%. Since it is easier, we will frame the problem by trying to figure out how many people need to be gathered before their chances of *not* sharing a birthday is less than 50%. This is an equivalent problem.

Imagine you are alone in a room. Out of all 365 days, you can have any birthday. People gradually start joining you. When the second person arrives, you calculate the probability that they do not share your birthday. Since there are 364 out of 365 days in the year that are not your birthday, the probability that they do not share your birthday is $\frac{364}{365}$.

When the third person arrives, the probability that they do not share a birthday with either person in the room is $\frac{363}{365}$ because two birthdays are already taken. When the fourth person arrives, the probability that they do not share any birthdays is $\frac{362}{365}$ and so on.

So how do you determine the probability that there are no shared birthdays overall in a group of people? When we combine the probabilities of independent events, we multiply their individual probabilities to get the total overall probability of something happening. For example, if the probability of rolling a four using a six-sided die is $1/6$ and the probability of picking a red marble out of a bag is $2/5$, then if you rolled a die and picked a marble, the probability that you would roll a four and retrieve a red marble is $1/6 \times 2/5 = 2/30 = 1/15$.

The same rule applies for the birthday problem. By multiplying the individual probabilities that each person does not share any birthdays of other people in the room, we can find the total probability that there are no shared birthdays in the room. We keep multiplying individual possibilities like this until we get a number that is less than 50% to figure out how many people need to be in a room before there is a greater than 50% chance that there is a shared birthday in the room. With 22 people:

$$\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \frac{361}{365} \times \frac{360}{365} \times \frac{359}{365} \times \frac{358}{365} \times \frac{357}{365} \times \frac{356}{365}$$
$$\times \frac{355}{365} \times \frac{354}{365} \times \frac{353}{365} \times \frac{352}{365} \times \frac{351}{365} \times \frac{350}{365} \times \frac{349}{365} \times \frac{348}{365} \times \frac{347}{365} \times \frac{346}{365}$$
$$\times \frac{345}{365} \times \frac{344}{365} = 52.5\%$$

The probability that you don't share a birthday is 52.5%. With the addition of the 23rd person, that value finally drops under 50%:

$$0.525 \times \frac{343}{365} = 49.3\%$$

Therefore, you need to have 23 people in a room before there is a greater than 50% chance that there will be a shared birthday. The old traveller was right! Since there are 23 people in the room including you, there is a greater than half chance that there is a shared birthday. For figuring out the proof, you rescue yourself and the group of people with you when you are teleported out of the room back into your own homes.

Proofs

A theorem is a mathematical statement that can be *proven* to be true. For example, the statement "if n and m can each be written as a sum of two perfect squares, so can their product $n \times m$" is a theorem. I can prove it by applying logical reasoning, definitions, and previously proven theorems to known facts. You've already seen one example of a proof: Euclid proved that there are an infinite number of prime numbers. On the first page, I showed that this theorem can be proven using logic, the definition of a prime number, and the Fundamental Theorem of Arithmetic, which is already proven to be true. By building on top of previous theorems like this, we can develop new mathematical understanding of the world that can later be built upon. Let's try working through a proof together.

Theorem:

If n and m can each be written as a sum of two perfect squares, so can their product $n \times m$.

Definition: a perfect square is a number x such that $x = y^2$ where y is any integer.
Distributive Law (a special case): $(u + v)(s + t) = (u \times s) + (u \times t) + (v \times s) + (v \times t)$
A Rule of Exponents: $(s^p \times t^p) = (s \times t)^p$

Proof:

Let $n = a^2 + b^2$ and $m = c^2 + d^2$, where $a, b, c,$ and d are integers. Then,

$$\begin{aligned} n \times m &= (a^2 + b^2)(c^2 + d^2) \\ &= (a^2 \times c^2) + (b^2 \times d^2) + (a^2 \times d^2) + (b^2 \times c^2) \\ &= (ac)^2 + (bd)^2 + (ad)^2 + (bc)^2 \\ &= (ac)^2 + 2abcd + (bd)^2 + (ad)^2 - 2abcd + (bc)^2 \\ &= (ac + bd)^2 + (ad - bc)^2 \end{aligned}$$

Therefore, $n \times m$ can be written as the sum of two perfect squares.

Take a moment to consider the reasoning that allowed me to move from one line to the next in the proof above.

There are many different ways to prove a theorem and depending on the situation, one way might be easier than another. For example, the Infinitude of Primes was proven on page 1 using a **Proof by Contradiction**. That is, I can show that statement A must be true by showing that its opposite, statement B, is impossible. The above proof is a **Direct Proof** and is the only kind of proof you need to worry about for now. You will learn more proof techniques in university.

Some More

I will also mention a few mathematical statements that I enjoy which you can either research on your own or wait until university to learn. Their proofs are beyond the scope of this lesson, so I will omit them.

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots=\infty$$

but

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots=0.693 \text{ (approximately)}$$

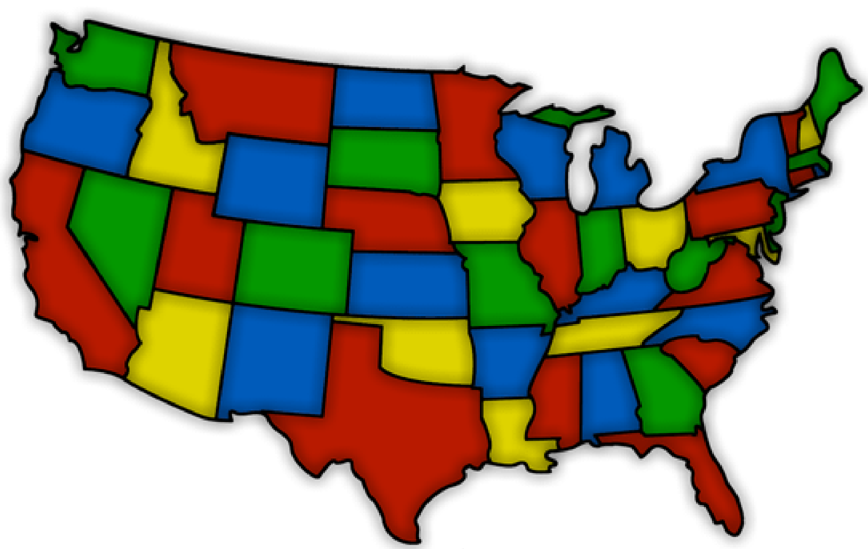
and

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots=\frac{\pi^2}{6}$$

The first sum is the harmonic series, the second is the alternating harmonic series and the third is known as the Basel problem.

The first major theorem to be proved using a computer was the **Four Colour Theorem**:

In order to colour any sort of graph that is separated into different regions, you only need to use four colours in order to ensure that none of the regions are the same colour as any region next to it.



a map of the US using four colours from mathigon

The Stable Marriage Problem: https://www.youtube.com/watch?v=Qcv1IqHWAzg
(and its mathematical proof): https://www.youtube.com/watch?v=LtTV6rIxhdo

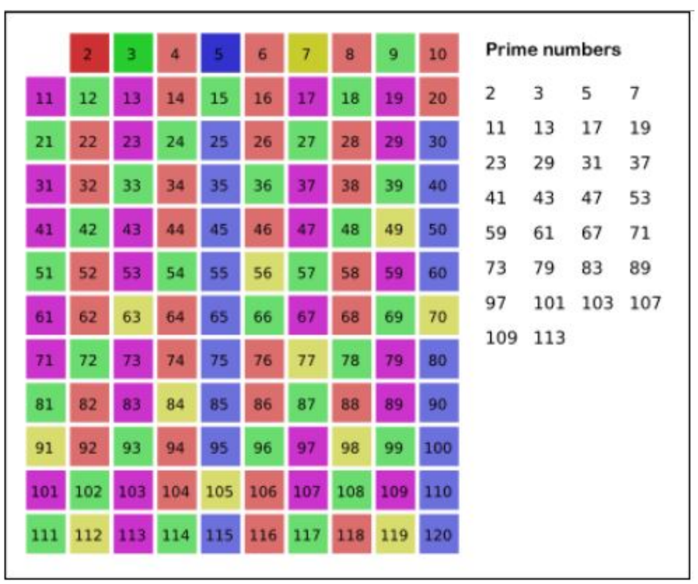
The Monty Hall Problem: https://www.youtube.com/watch?v=4Lb-6rxZxx0

The Prisoner's Dilemma (Game Theory): https://www.youtube.com/watch?v=t9Lo2fgxWHw

Problem Set

1. (a) If E is an event that will certainly happen, its probability $P(E)$ is 1. If an event is impossible, then its probability is 0. The opposite of event E is $\overline{E}$. The probability of the opposite of an event happening is then $P(\overline{E})$. Determine a formula to calculate $P(\overline{E})$ in terms of $P(E)$. Remember: the probabilities of most events are neither certain or impossible. $P(\overline{E}) = 1 - P(E)$

   (b) Trying to predict how often prime numbers occur is notoriously hard because primes seem to appear randomly in a number set. However, it is possible to find all primes in a given set. Using the figure below, determine an algorithm for finding all the primes under 120. Write down the primes you find on the side under "Prime Numbers": (See the animation here: Sieve of Eratosthenes)



One method is to cross out the multiples of all primes you encounter until all you are left with are numbers which have no positive factors other than one and themselves. It is efficient to only check up to the square root of the largest number. In this case, $\sqrt{120} < 11$ (the closest prime) so we only have to cross out the multiples of primes under 11. All remaining numbers will be prime. There are many algorithms to find primes, and this is only one of them.

(c) Based on your results to part (b), solve the following: if you were to randomly place your pencil on the figure and later look at the number on the square you landed on, what is the probability

i. that you will land on 3 prime numbers in three consecutive turns?

There are 30 prime numbers to choose from out of 119 numbers, so the total probability is $\left(\frac{30}{119}\right)^3 = 1.6\%$

ii. that you will land on the *same* prime number in three consecutive turns?

You start off with 30 primes to choose from. After your first turn, you can only land on the same number again so there is only 1 option out of 119 for the remaining turns. The total probability is therefore $\frac{30}{119^3} = 0.00178\%$

iii. that you will land on three *different* prime numbers in three consecutive turns?

We start off with 30 prime numbers to choose from. After landing on a prime number your first turn, the number of primes left that you can land on goes down by one. The total probability is $\frac{30 \times 29 \times 28}{119^3} = 1.44\%$

* iv. that the following situation will *not* happen?

You take 40 turns. On every even numbered turn, you land on a <u>new</u> prime number (you haven't landed on that prime number on any previous terms). On every odd numbered term, you land on a composite number.

First, the probability that it would happen: There are 20 odd numbered turns and you can get any of 89 composite numbers on those turns out of 119 numbers in total: $\frac{89^{20}}{119^{20}}$. On the remaining 20 turns, you start off with 30 primes to choose from and the choices for primes decreases by one until on the 40th turn there are only 11 primes to choose from out of 119 total numbers: $\frac{30 \times 29 \times 28 \times \ldots \times 11}{119^{20}}$. Multiply these two probabilities together to get the total probability

$$\frac{89^{20}}{119^{20}} \times \frac{30 \times 29 \times 28 \times \ldots \times 11}{119^{20}} = 0.00000000000000000006758$$

Now, the probability that this will *not* happen is:

$$1 - 0.00000000000000000006758 = 0.99999999999999999993242 \approx 100\%$$

2. If there are 10 people in a room, what is the probability that two people are born in the same month? The same week?
   Same month:
   This is just like the birthday paradox. The probability that there is a shared birthday month is going to be 1 – the probability that there is not a shared birthday month:

$$1 - \frac{12 \times 11 \times 10 \ldots \times 3}{12^{10}} = 99.6\%$$

   Same week:
   The probability that there is a shared birthday week is going to be 1 – the probability that there is not a shared birthday week, keeping in mind that there are 52 weeks in a year:

$$1 - \frac{52 \times 51 \times 50 \ldots \times 43}{52^{10}} = 60.3\%$$

3. (a) Let's say you are in a very important business meeting. There are 10 people in the room, including you. Everyone starts shaking hands to introduce themselves to each other. You can't shake hands with yourself and you only ever shake hands with another person once (after all, what's the point of reintroducing yourself?). How many total handshakes will there be?
   The first person shakes hands with 9 people (Person 1 = 9 new handshakes), then the second person shakes hands with 9 people minus the one handshake we already counted with the first person (Person 2 = 8 new handshakes). The third person shakes hands with 9 people minus the handshakes we already counted with the first and second people (Person 3 = 7 new handshakes). Following this pattern, we find that there are 45 handshakes in total.

* (b) If there are n people in the room, how many total handshakes will there be?

If there are n people in a group, the total number of handshakes will be $1 + 2 + 3 + 4 + \ldots + (n-1)$ according to the pattern we noticed in part (a). Going back to our solution there, the answer was $1+2+3+4+5+6+7+8+9$ handshakes. We can group the terms of this sum together so that each grouping is equal to 9: $(1+8)+(2+7)+(3+6)+(4+5)+9 = 5 \times 9$. There are $n/2$ or 5 total groupings and each grouping is equal to $n-1$ or 9. That is, there are $\frac{n(n-1)}{2}$ handshakes for n people.

* 4. (a) What is the definition of an even number and how can you write it algebraically (using numbers and variables)?

Any even number x can be written in the form $x = 2k$, where k is any integer.

(b) What is the definition of an odd number and how can you write it algebraically?

Any odd number y can be written in the form $y = 2k + 1$, where k is any integer.

(c) Prove that the sum of any two even integers x and y is even.

Let $x = 2t$ and $y = 2s$ where s, t are integers. We analyze their sum:

$$\begin{aligned} x + y &= 2t + 2s \\ &= 2(t + s) \end{aligned}$$

Since $t+s$ is an integer, we have written the sum $x+y$ in the form $x+y = 2k$ where k is any integer. Therefore we have shown that $x + y$ is always even according to the definition of an even number.

(d) If x is even and y is odd, prove that $x \times y$ is even.

Let $x = 2t$ and $y = 2s + 1$ where s, t are integers.

$$\begin{aligned} x \times y &= (2t)(2s + 1) \\ &= 4ts + 2t \\ &= 2(2ts + t) \end{aligned}$$

$2ts + t$ is an integer (multiplying and adding whole numbers gives you another whole number), so we have written the product $x \times y$ in the form $2k$ where k is an integer. Therefore, $x \times y$ is always even.

(e) Prove that if x is an even number, so is x^2.

Let $x = 2t$ where t is an integer.

$$\begin{aligned} x^2 &= (2t)(2t) \\ &= 4t^2 \\ &= 2(2t^2) \end{aligned}$$

$2t^2$ is an integer, so x^2 is even.

(f) Prove that if y is an odd number, so is y^2.

Let $y = 2s + 1$ where s is an integer.

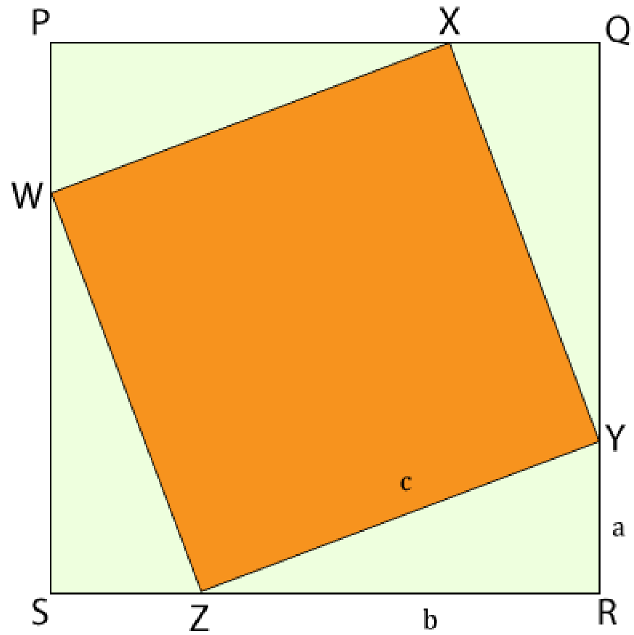
$$\begin{aligned} y^2 &= (2s+1)(2s+1) \\ &= 4s^2 + 4s + 1 \\ &= 2(2s^2 + 2s) + 1 \end{aligned}$$

Since $2s^2 + 2s$ is an integer, we have shown that y^2 is odd according to the definition of an odd number.

* 5. Prove the Pythagorean theorem given the distributive law and the following diagram: The square WXYZ with side length c is drawn inside the square PQRS.

- Point X lies on the line $\overline{PQ}$.
- Point Y lies on the line $\overline{QR}$.
- Point Z lies on the line $\overline{RS}$.
- Point W lies on the line $\overline{SP}$.
- a is the length of the line $\overline{RY}$.
- b is the length of the line $\overline{ZR}$.

hint: Use All Real Evidence of Aliens. Use AREA



* denotes a difficult question

Notice that since WXYZ is a square, its sidelength c is the same all around. This means that all of the triangles in the picture must have the same base, height, and hypoteneuse length. Therefore, the sidelength of the outer square must be $a + b$.

The area of the bigger square is equal to the area of the smaller square plus the areas of the four triangles. We know that the area of the bigger square is $(a+b)^2$, the area of the smaller square is c^2 and the area of each of the four triangles is $\frac{1}{2}ab$:

$$(a + b)^2 = c^2 + 4(\frac{1}{2}ab)$$
$$a^2 + 2ab + b^2 = c^2 + 2ab$$
$$a^2 + b^2 = c^2$$