



Grade 7/8 Math Circles

March 2 & 3 2016

Word Problems

Although they can be confusing, you can tame word problems by organizing and modeling the problem. There are several different types of word problems and each one comes with its own solution technique. One type of word problem called the “Logic Puzzle” was covered in the first week of this term. Today, we are going to talk about the kinds of word problems that can be solved using a system of mathematical equations.

Before beginning a problem, write down what information the problem has given you (**knowns**) and what information it is asking you to find (**unknowns**). We use variables to represent unknown values. Then, use the context of the question to relate your knowns and unknowns through a mathematical equation. Sometimes, you are expected to already know these equations (for example: formulas for area and perimeter of various shapes) and other times you will have to construct an equation yourself to fit the problem. Now all you have to do is solve your system of equations for your unknown(s). Today, we will explore how we can use **substitution** to solve a system of equations.

Example 1

A large box of chocolates and a small box of chocolates together costs \$15. If the large box costs twice as much as the small box, what are the individual prices of the two boxes? (Gauss 2007 [†])

Knowns:

Total cost = \$15

Unknowns:

Let the price of the large box in dollars be L .

Let the price of the small box in dollars be S .

Equations:

From the question: the total cost results from buying one large box and one small box:

$$L + S = 15$$

The big box is twice the size of the small box, so we could buy two small boxes for the price of a big box:

$$L = 2 \times S$$

Now we have a system of equations:

$$L + S = 15 \tag{1}$$

$$L = 2 \times S \tag{2}$$

Solve one of the equations for any variable and “plug” in the expression on the other side into a different equation. Then, solve the new equation. This is called the substitution method and we can use it to solve this system of equations to get the value of our two unknowns. We can see in our system that equation (2) is already solved for the variable L . We plug in to expression on the other side of equation (2) into L in equation (1):

$$L + S = 15$$

$$(2 \times S) + S = 15$$

$$3 \times S = 15$$

$$S = 5$$

I used brackets when I plugged in L (second step) to keep track of my substitution and to preserve BEDMAS. Always do this!

I now plug $S = 5$ back into equation (2) to solve for L :

$$L = 2 \times S$$

$$L = 2 \times 5$$

$$L = 10$$

Therefore, the price of the large box is \$10 and the price of the small box is \$5.

You should always end word problems with a therefore statement to show that you’ve actually solved the problem. If the question is in words, your answer should be too!

Example 2

The sum of three consecutive odd numbers is 57. What is the product of these three numbers? (Gauss 2014 †)

Knowns:

$$\text{Sum} = 57$$

Unknowns:

Let the smallest number be S .

Let the middle number be M .

Let the largest number be L .

Equations:

Let's do the easy one first:

$$S + M + L = 75$$

We usually denote an even number like this: $2n$, and an odd number like this: $2n + 1$ where n is any number greater than or equal to zero. So let's say:

$$S = 2n + 1$$

$$M = 2n + 3$$

$$L = 2n + 5$$

Notice that I could come up with these formulas because I knew that consecutive odd numbers are two apart.

Now, I'm going to plug in my expressions for S , M , and L into my sum equation to relate it to my known.

$$S + M + L = 57$$

$$(2n + 1) + (2n + 3) + (2n + 5) = 57$$

$$6n + 9 = 57$$

$$6n = 48$$

$$n = 8$$

I can plug this value for n back into my equations for $S, M,$ and L to solve for the three unknowns.

$$S = 2n + 1 = 2(8) + 1 = 17$$

$$M = 2n + 3 = 2(8) + 3 = 19$$

$$L = 2n + 5 = 2(8) + 5 = 21$$

My three consecutive odd numbers are 17, 19, and 21. I have to find their product:

$$17 \times 19 \times 21 = 6783$$

Therefore, the product of the three numbers is 6783.

Theoretical note:

In the problem above, we had 4 unique equations and 4 unknowns ($S, M, L,$ and n) in our system of equations. Since none of the equations are linear combinations of any others, we were able to find the values of all four of those unknowns. If I plugged these values back into the equations, each equation would be satisfied (left hand side = right hand side). This is called a **determined** system of equations. In an **overdetermined** system, there are more equations than unknowns and in an **underdetermined** system, there are fewer equations than unknowns. Overdetermined systems have no solutions (values for your unknowns can't satisfy all of the equations), unless some of the equations are linear combinations of the others. An underdetermined system may have no solutions or an infinite number of solutions (some of your unknowns can have any value).

Example 3

The width of a rectangle is doubled and the length is halved. This produces a square with a perimeter of P . What is the perimeter of the original rectangle? (Gauss, 2015)

Knowns:

(no values)

Unknowns:

Let the perimeter of the square be P .

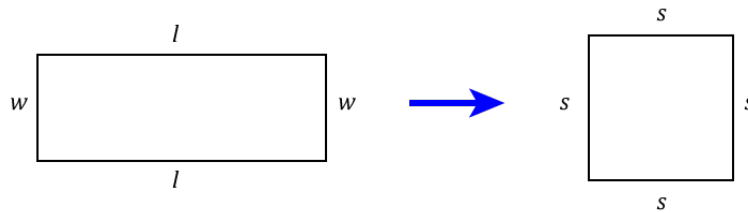
Let the perimeter of the rectangle be R .

Let the sides of the square have length s .

Let the width of the rectangle be w .

Let the length of the rectangle be l .

Drawing diagrams is especially useful for questions involving geometry.



Equations:

The perimeter P of a square with side length s is:

$$P = 4s$$

The perimeter R of a rectangle with length l and width w is:

$$R = 2l + 2w$$

Since the length of the square's sides are a direct result of changing the width and length of the rectangle, we can say:

$$s = 2w$$

$$2s = l$$

This is an underdetermined system of equations: that is, there are fewer equations than unknowns. This means that some of my unknowns will remain unsolved.

Let's plug in the third and fourth equation into the second equation, since we are trying to solve for R :

$$R = 2l + 2w$$

$$R = 2(2s) + (s)$$

$$R = 4s + s$$

$$R = 5s$$

Now, we have solved for the rectangle's perimeter in terms of s . However, the only unknown mentioned by name in the question is P , so we have to give the answer in terms of P . This is good practice because you have defined the other variables yourself so they don't mean anything to the person asking the question.

I can solve the first equation for s and substitute the resulting expression:

$$P = 4s$$

$$\implies s = \frac{1}{4} \times P$$

$$\implies R = 5s$$

$$= 5 \times \left(\frac{1}{4} \times P \right)$$

$$= \frac{5}{4}P$$

Therefore, the perimeter of the original rectangle is $\frac{5}{4}P$.

Having variables in the solution is often unavoidable with underdetermined systems. P can have any value. If we were given the values for some of the variables in this system, we would have fewer unknowns and could reduce this to a determined system. Then we would have a definite value for P (and for all the other variables, too).

Exercise: Suppose that the original rectangle is 5 units wide. What are the perimeters of the square and rectangle?

$$s = 2w = 2 \times 5 = 10$$

$$P = 4s = 4 \times 10 = 40$$

$$R = \frac{5}{4}P = \frac{5}{4} \times 40 = 50$$

Therefore, the square's perimeter is 40 units and the rectangle's perimeter is 50 units.

Problem Set

1. Hannah scored 312 points during the basketball season. If her average (mean) was 13 points per game, how many games did she play? (Gauss, 2015)

The average (mean) number of points scored per game multiplied by the number of games played is equal to the total number of points scored during the season. Therefore, the number of games that Hannah played is equal to the total number of points she scored during the season divided by her average (mean) number of points scored per game, or $312 \div 13 = 24$.

2. The sum of four numbers is T . Suppose that each of the four numbers is now increased by 1. These four new numbers are added together and then the sum is tripled. What is the value of this final result? (Gauss, 2011)

If each of the four numbers is increased by 1, then the increase in their sum is 4. That is, these four new numbers when added together have a sum that is 4 more than their previous sum T , or $T + 4$. This new sum $T + 4$ is now tripled. The result is $3 \times (T + 4) = (T + 4) + (T + 4) + (T + 4) = 3T + 12$.

3. To rent a kayak and a paddle, there is a fixed fee to use the paddle, plus a charge of \$5 per hour to use the kayak. For a three hour rental, the total cost is \$30. What is the total cost for a six hour rental? (Gauss, 2007)

The total cost to use the kayak for 3 hours is $3 \times \$5 = \15 . Since the total rental cost for 3 hours is \$30, then the fixed fee to use the paddle is $\$30 - \$15 = \$15$. For a six hour rental, the total cost is thus $\$15 + (6 \times \$5) = \$15 + \$30 = \$45$.

4. Together, Akira and Jamie weigh 101 kg. Together, Akira and Rabia weigh 91 kg. Together, Rabia and Jamie weigh 88 kg. How many kilograms does Akira weigh? (Pascal, 2002)

Since Akira and Jamie together weigh 101 kg, and Akira and Rabia together weigh 91 kg, then Jamie weighs 10 kg more than Rabia. So let Rabia's weight be x kg. Then Jamie's weight is $(10 + x)$ kg. So from the third piece of information given,

$$x + x + 10 = 88$$

$$2x = 78$$

$$x = 39$$

Since Rabia weighs 39 kg, Akira weighs 52 kg, since their combined weight is 91 kg.

5. School B has 240 students, of whom 30% received a ride. How many more of the 240 students in School B needed to receive a ride so that 50% of the students in School B got a ride? (Galois, 2015)

Solution 1

At School B, 30% of 240 students or $\frac{30}{100} \times 240 = 7200/100 = 72$ students received a ride. If 50% or $1/2$ of the students in School B were to receive a ride, then $1/2 \times 240 = 120$ students would get a ride. Therefore, $120 - 72 = 48$ more students needed to receive a ride so that 50% of the students in School B got a ride.

Solution 2

As a percent, the difference between 50% of students receiving a ride and the 30% of students who did receive a ride is 20%. Therefore, 20% of 240 students or $\frac{20}{100} \times 240 = 4800/100 = 48$ additional students would need to receive a ride so that 50% of students in School B got a ride.

6. The Emus won 4 of their first 10 games. The team played x more games and won all of these. Their final winning percentage was 70%. How many games did they play in total? (Fryer, 2008)

The Emus played $10 + x$ games in total and won $4 + x$ games in total. Since their final winning percentage was 70%, then $\frac{4 + x}{10 + x} \times 100\% = 70\%$ or $\frac{4 + x}{10 + x} = \frac{7}{10}$.

Therefore, $10 \times (4 + x) = 7 \times (10 + x)$, so $40 + 10x = 70 + 7x$ or $3x = 30$ or $x = 10$. Therefore, the Emus played $10 + x = 10 + 10 = 20$ games in total.

7. At Barker High School, a total of 36 students are on either the baseball team, the hockey team, or both. If there are 25 students on the baseball team and 19 students on the hockey team, how many students play both sports? (Fermat, 2015)

Solution 1

The two teams include a total of $25 + 19 = 44$ players. There are exactly 36 students who are at least one team. Thus, there are $44 - 36 = 8$ students who are counted twice. Therefore, there are 8 students who play both baseball and hockey.

Solution 2

Suppose that there are x students who play both baseball and hockey. Since there are 25 students who play baseball, then $25 - x$ of these play baseball and not hockey. Since there are 19 students who play hockey, then $19 - x$ of these play hockey and not

baseball. Since 36 students play either baseball or hockey or both, then

$$(25 - x) + (19 - x) + x = 36$$

(The left side is the sum of the numbers of those who play baseball and not hockey, those who play hockey and not baseball, and those who play both.)

Therefore, $44 - x = 36$ and so $x = 44 - 36 = 8$. Thus, 8 students play both baseball and hockey.

- * 8. At Cornthwaite H.S., many students enroll in an after-school arts program. The program offers a drama class and a music class. Each student enrolled in the program is in one class or both classes. In 2014, a total of 80 students enrolled in the program. Let x represent the number of students in both classes. If there were five fewer than thrice x students in the drama class, and thirteen greater than six times x students in the music class, how many students were in both classes?

(Canadian Intermediate Math Contest, 2015 †)

The number of students in the drama class is $3x - 5$ and the number of students in the music class is $6x + 13$. Together, the number of students in both classes should add up to the total number of students enrolled, 80. However, the number of students in both classes appear in both the equations above. In order to prevent double counting them, we have to subtract the total number of students in both classes, x , from one of the equations above. Thus:

$$(3x - 5 - x) + (6x + 13) = 80$$

$$(2x - 5) + (6x + 13) = 80$$

$$8x + 8 = 80$$

$$8x = 72$$

$$x = 9$$

Therefore, there are 9 students enrolled in both classes.

For more practice questions, see: http://cemc.uwaterloo.ca/contests/past_contests.html

† indicates a modified problem

* indicates a difficult question