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Grade 7/8 Math Circles

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Review

Logic

1. The word MATHEMATICS is to be centered in a space allowing 37 letters. No spaces are permitted between the letters. How many blank spaces must the typist leave before starting to type the word?

   MATHEMATICS has 11 letters, so first we calculate: $37 - 11 = 26$, which is the number of spaces surrounding the word.

   Since we want it to be centered, we divide these 26 spaces in 2 equal parts, so there will be 13 spaces before, and 13 spaces following the word MATHEMATICS.

2. Each letter in the subtraction below represents a single digit. What are k, p, q, and r?

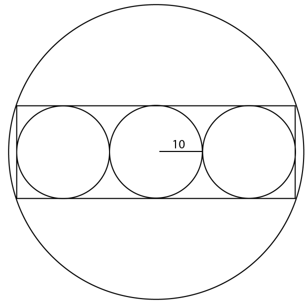
$$\begin{array}{rcccc} & 6 & p & q & r \\ - & k & 3 & 5 & 9 \\ \hline & 1 & 5 & 8 & 8 \end{array}$$

   k = 5, p = 9, q = 4, and r = 7

3. At a party, exactly 15 people ate hot dogs, 12 ate hamburgers. 10 people ate both, and 3 ate neither. How many people were at the party?

   We must add the number of people who did not eat, those who ate hot dogs and those who ate hamburgers, but since 10 ate both hamburgers and hot dogs, that means they were counted twice. Thus there are $15 + 12 - 10 + 3 = 20$

4. In the diagram below, what is the area of the largest circle?



If you consider a right angle triangle from the center of the circle to the top left corner of the rectangle embedded in the large circle, you will have a right angle triangle with $b = 30$ and $h = 10$. Using Pythagorean Theorem, This tells us that the hypotenuse, which is also the radius of the large circle, is $r^2 = 10^2 + 30^2 = 100 + 900 = 1000$. Therefore, the area of the large circle is

$$A = \pi r^2 = 1000\pi$$

5. A positive integer is to be placed in each box. The product of any 4 adjacent integers is 120. What is the value of x?

		2			4			x			3		

Since the product of any four integers is 120, $a_1a_2a_3a_4 = a_2a_3a_4a_5 = 120$ where a_n represents the number in the n^th box. Therefore, $a_1 = a_5$ and similarily $a_2 = a_6, a_3 = a_7, a_4 = a_8$ or more generally, $a_n = a_{n+4}$. The boxes can be filled as follows:

x	4	2	3	x	4	2	3	x	4	2	3	x	4

$$\text{Therefore, } (4)(2)(3)(x) = 120$$
$$x = \frac{120}{24} = 5$$

Modular Arithmetic

1. What is $604 \times 123 \pmod{60}$? 12

2. Convert 1110001 to decimal.
   $(1\times2^6)+(1\times2^5)+(1\times2^4)+(0\times2^3)+(0\times2^2)+(0\times2^1)+(1\times2^0) = 64+32+16+1 = 113$

3. Convert 341 to binary.

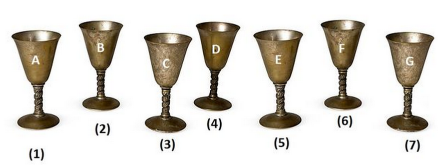
$$\begin{aligned}
341 &= 2(170) + 1\\
170 &= 2(85) + 0\\
85 &= 2(42) + 1\\
42 &= 2(21) + 0\\
21 &= 2(10) + 1\\
10 &= 2(5) + 0\\
5 &= 2(2) + 1\\
2 &= 2(1) + 0\\
1 &= 2(0) + 1\\
&\Rightarrow 101010101
\end{aligned}$$

4. Anna was facing East and rotated 3195° counterclockwise. Which direction is she facing now?

$$3195 \pmod{360} = 315$$

   Therefore, Anna rotated 315° counterclockwise from East.
   Therefore she is now facing SE.

5. You have 7 goblets one of which is real gold. When you align them and count (back and forth starting with A, B, C, D, E, F, G, F, E, D,) then the golden goblet would be the 1000th one that you count. Which one is the golden goblet?



   1 full round $(A, B, C, D, E, F, G, F, E, D, C, B)$ has 12 goblets. So $1000 \pmod{12} = 4$. Therefore the 4^{th} goblet made of real gold.

Number Theory

1. How many positive integers less than 20 are co-prime with 20? What are those numbers?

   There are 8 numbers: 1, 3, 7, 9, 11, 13, 17, and 19.

2. What is the GCD of 861 and 984?

$$
\begin{aligned}
984 &\pmod{861} = 123 \\
861 &\pmod{123} = 0 \\
\therefore \gcd(984, 861) &= 123
\end{aligned}
$$

3. What is the remainder when 2^{2007} is divided by 15?

   If we consider the first few $2^n \pmod{15}$, we will notice a pattern.

$$
\begin{aligned}
2^0 &\pmod{15} = 1 \\
2^1 &\pmod{15} = 2 \\
2^2 &\pmod{15} = 4 \\
2^3 &\pmod{15} = 8 \\
2^4 &\pmod{15} = 1 \\
2^5 &\pmod{15} = 2 \\
2^6 &\pmod{15} = 4 \\
2^7 &\pmod{15} = 8 \\
&\vdots \\
2^{2004} &\pmod{15} = 1 \\
2^{2005} &\pmod{15} = 2 \\
2^{2006} &\pmod{15} = 4 \\
2^{2007} &\pmod{15} = 8
\end{aligned}
$$

4. What is the remainder when 5^{119} is divided by 59?

   Solution 1:

   We will use Fermat's Little Theorem (Part I), which state that for any prime number p and any positive integer a not divisible by p, $a^{p-1} \pmod{p} = 1$.

$$\begin{aligned}
&5^{119} \pmod{59} \\
&= 5^{(88\times 2)+3} \pmod{59} \\
&= (5^{88})^2 \times 5^3 \pmod{59} \\
&= (1)^2 \times 125 \pmod{59} \\
&= 7
\end{aligned}$$

   Solution 2:

   We can also use Fermat's Little Theorem (Part II), which state that for any prime number p and any positive integer a, $a^p \pmod{p} = a$.

$$\begin{aligned}
&5^{119} \pmod{59} \\
&= (5^{59})^2 \times 5^1 \pmod{59} \\
&= (5)^2 \times 5 \pmod{59} \\
&= 5^3 \pmod{59} \\
&= 7 \pmod{59}
\end{aligned}$$

   Therefore, the remainder when 5^{119} is divided by 59 is 7.

5. The number of students in a school is an integer between 500 and 600. When grouped into groups of 12, 20 or 36, there are 7 students left over. How many students are in the school?

   To solve this problem we will write down all the multiples of 12, 20 and 36 (individually) between 500 and 600.

   Multiples of 12: 504, 516, 528, 540, 552, 564, 576, 588, 600.

   Multiples of 20: 500, 520, 540, 560, 580, 600.

   Multiples of 36: 504, 540, 576.

   The only common multiple is 540, but since there are 7 left over students when divided by either 12, 20 or 36, then there must by 547 students in the school.

Word Problems

1. A set of 5 different positive integers has an average of 11. What is the largest possible number in this set? (Guass, 2000).
   If the set of five different positive integers has an average of 11, the five integers must sum to 5×11 or 55. The four smallest possible integers are 1, 2, 3, and 4. The largest possible integer in the set is $55 - (1 + 2 + 3 + 4) = 45$.

2. In a certain month, three of the Sundays have dates that are even numbers. Which day of the week does the tenth of this month fall on? (Gauss, 2000).
   A Sunday must occur during the first three days of any month with five Sundays. Since it is on an even day, it must be on the second day of the month. This implies that the ninth day of the month is also a Sunday, which makes the tenth day a Monday.

3. Water is poured from a full 1.5 L bottle into an empty glass until both the glass and the bottle are 3/4 full. What is the volume of the glass? (Gauss, 2004)
   Solution 1
   When the pouring stops, $1/4$ of the water in the bottle has been transferred to the glass. This represents $3/4$ of the volume of the glass. Therefore, the volume of the bottle is three times the volume of the glass, so the volume of the glass is 0.5 L.

   Solution 2
   When the pouring stops, $1/4$ of the water in the bottle or $1/4 \times 1.5 = 0.375$ L of water is in the glass. Since this represents $3/4$ of the volume of the glass, then the volume of the glass is $4/3 \times 0.375 = 0.5$ L.

4. What is $a + b$ if $a(x + b) = 3x + 12$ is true for all values of x? (Cayley, 2013).
   Solution 1
   Since $a(x + b) = 3x + 12$ for all x, then $ax + ab = 3x + 12$ for all x.
   Because the equation is true for all x, the coefficients on the left side must match the coefficients on the right side.
   Therefore, $a = 3$ and $ab = 12$ which gives $3b = 12$ or $b = 4$.
   Finally, $a + b = 3 + 4 = 7$.

   Solution 2
   Since $a(x + b) = 3x + 12$ for all x, then the equation is true for $x = 0$ and $x = 1$.
   When $x = 0$, we obtain $a(0 + b) = 3(0) + 12$ or $ab = 12$.

When $x = 1$, we obtain $a(1 + b) = 3(1) + 12$ or $a + ab = 15$.
Since $ab = 12$, then $a + 12 = 15$ or $a = 3$.
Since $ab = 12$ and $a = 3$, then $b = 4$.
Finally, $a + b = 3 + 4 = 7$.

5. $109/x$ leaves a remainder of 4. What is the sum of all such two digit positive integers x? (Cayley, 2013).
   Suppose that the quotient of the division of 109 by x is q.
   Since the remainder is 4, this is equivalent to $109 = qx + 4$ or $qx = 105$.
   Put another way, x must be a positive integer divisor of 105.
   Since $105 = 5 \times 21 = 5 \times 3 \times 7$, its positive integer divisors are

$$1,\ 3,\ 5,\ 7,\ 15,\ 21,\ 35,\ 105$$

   Of these, 15, 21, and 35 are two digit positive integers so are the possible values of x.
   The sum of these values is $15 + 21 + 35 = 71$.

Physics

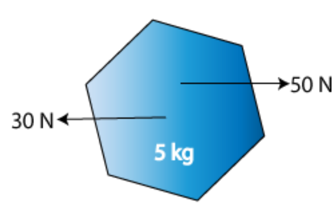
1. What is the difference between speed and velocity?
   Speed is a scalar (has a size) and velocity is a vector (has both a size and a direction).
   Furthermore, speed is distance / time and velocity is displacement / time.

2. What quantity does the following formula represent and what are its units?

$$\frac{v_2 - v_1}{t}$$

   Average accleration, with units $\frac{m}{s^2}$

3. What is the acceleration of the block? What is the acceleration if the left force is 50 N?



The unbalanced force is

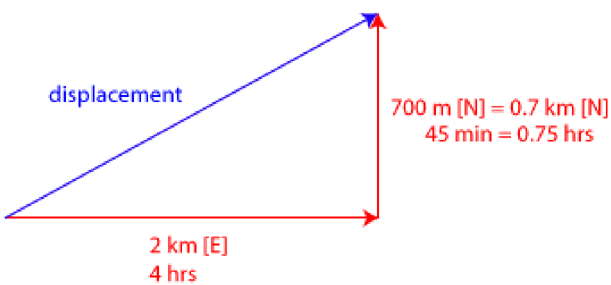
$$\begin{aligned} F &= 50 \text{ N [right]} + 30 \text{ N [left]} \\ &= 50 \text{ N [right]} - 30 \text{ N [right]} \\ &= 20 \text{ N [right]} \end{aligned}$$

Using Newton's second law $F = ma$, we get that the acceleration is

$$\begin{aligned} a &= \frac{F}{m} \\ &= \frac{20 \text{ N [right]}}{5 \text{ kg}} \\ &= 4 \ \frac{\text{m}}{\text{s}^2} \text{ [right]} \end{aligned}$$

If instead the left force was 50 N, there would be zero unbalanced force which means that the acceleration of the block in that case would be $0 \ \frac{\text{m}}{\text{s}^2}$.

4. I travel 2 km [E] in 4 hrs and 700 m [N] in 45 mins. What is my velocity and speed? Give 2 decimal places.



First, we have to convert all our quantities to the same units. I chose to convert my distance quantities to km and my time quantities to hours.

Total time = 4 hrs + 0.75 hrs = 4.75 hrs

Total distance = 2 km + 0.7 km = 2.7 km

Total displacement (we need the Pythagorean theorem):

$$\begin{aligned} (\text{displacement})^2 &= (2 \text{ km})^2 + (0.7 \text{ km})^2 \\ &= 4 \text{ km}^2 + 0.49 \text{ km}^2 \\ &= 4.49 \text{ km}^2 \end{aligned}$$

Therefore:

displacement = $\sqrt{4.49 \text{ km}^2}$ = 2.12 km [NE] (direction obtained from diagram)

Velocity is displacement / time:

Velocity = (2.12 km [NE]) / (4.75 hrs) = $0.45 \ \frac{\text{km}}{\text{hr}}$ [NE]

Speed, on the other hand, is distance / time:

Speed = (2.7 km) / (4.75 hrs) = $0.57 \ \frac{\text{km}}{\text{hr}}$

5. When is size of velocity = speed?

When size of displacement = distance (movement in a straight line).

6. How much would you weigh on Jupiter, where acceleration due to gravity is 2.65g? g is the acceleration due to gravity on Earth. Assume your mass is 42 kg.
   g $= 9.8\,\frac{m}{s^2}$. From $F = ma$, we find that a person with mass 42 kg would weigh $(9.8)(2.65)(42) = 1091$ N on Jupiter.

7. What is the weakest fundamental force?
   Gravitation

Mathematical Thinking

1. How many primes are there?
   Infinitely many. See Euclid's proof of the Infinitude of Primes.

2. What is the probability that I will not: roll a six and then something other than a six on a six-sided die? (in %).
   First, we find the probability that the above event *will* happen. The probability of rolling a six on a six sided die is 1/6. The possibility of rolling anything other than a six is 5/6. Multiplying these independent probabilities together, we get the total probability of rolling a 6 and then something other than a 6 is

   $$\frac{1}{6} \times \frac{5}{6} = 14\%$$

   Now, the probability that this will *not* happen must be $100\% - 14\% = 86\%$ since there is a 100% chance that this event either will or will not happen.

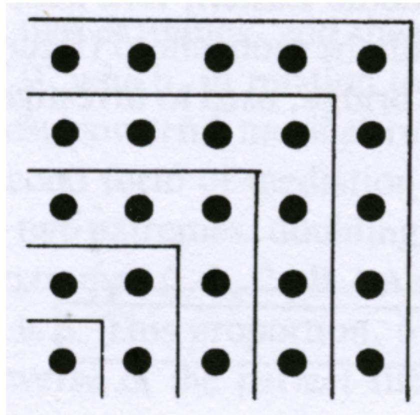
   So the answer is 86%.

3. Prove that the product of two odd numbers is odd.
   Let $x = 2k + 1$ and $y = 2t + 1$ where k, t are integers.

   $$\begin{aligned} xy &= (2k+1)(2t+1) \\ &= 4kt + 2k + 2t + 1 \\ &= 2(2kt + k + t) + 1 \end{aligned}$$

   Since $2kt + k + t$ is an integer, xy is odd by definition of an odd number.

4. This image shows that $1 + 3 + 5 + 7 + 9 = 5 \times 5$. What is $1 + 3 + 5 + \ldots + 99$?



The sum of the first n odd integers is n^2. Therefore, $1+3+5+\ldots+99 = 50 \times 50 = 2500$

5. Prove that $\sqrt{2}$ cannot be written as an irreducible fraction of two whole numbers (ie. it is irrational).

Assume the opposite. That is, assume $\sqrt{2}$ is rational and therefore can be written as a ratio of the integers a and b.

$$\sqrt{2} = \frac{a}{b}$$

The fraction on the right is irreducible (in lowest terms). This means that a and b are coprime (have no factors in common). We can write any rational number as an irreducible fraction like this. We rearrange the above equation for a:

$$2 = \frac{a^2}{b^2}$$

$$a^2 = 2b^2$$

From the mathematical thinking problem set, we know that this means a^2 is even and therefore a is even. We can therefore write $a = 2k$ where k is any integer.

$$2b^2 = (2k)^2$$
$$2b^2 = 4k^2$$
$$b^2 = 2k^2$$

The last line implies that b^2 (and therefore b itself) is even. This is a contradiction because in our assumption that $\sqrt{2}$ was rational, we said that we could write it as the irreducible fraction of two whole numbers. If a and b are both even, the fraction $\frac{a}{b}$ can be reduced. Therefore, our assumption that $\sqrt{2}$ is rational must be wrong (it's the only assumption we made). So we have proven that $\sqrt{2}$ is irrational. That is, it can't be written as a fraction of two whole numbers.