



Grade 7/8 Math Circles

March 8 & 9 2016

Physics

Physics is the study of how the universe behaves. This includes everything from the movement of planets and galaxies to the interactions between subatomic particles. In order to decipher the relationships between physical objects, we need to use math to describe and explain what happens around us. Today, we will learn how to do this by exploring a specific branch of physics called **mechanics** that tries to describe why and how things move.

Vectors and Scalars

Size is used to describe how big something is. So which would you say is “bigger”: 4 km or 18 km? 15 seconds or 15 minutes? The speed of a bike or the speed of a train? All of these things have a property that we call “size” or “magnitude”. That is, they are quantifiable: we can assign a number to them and compare that quantity to other quantifiable things. 15 minutes is longer or “bigger” than 15 seconds, in a sense. Let’s try something different: which quantity is “bigger”; 5 m to the left or 5 m to the right? Trick question! They both have the same size, but they *are* different – they have different *directions*.

A **vector** is a quantity with both a size and a direction, whereas a **scalar** is a quantity with only a size. For example, 5 m would be a scalar quantity because it has a size associated with it – 5 m is 5 m long. On the other hand, 5 m to the right is a vector quantity because there is both a size and a direction associated with it.

Displacement

The term **displacement** refers to the separation between a starting point and an ending point. For example, if I stand at point A and walk for 5 meters in a straight line to point C, then the size of my displacement is 5 meters (Figure 1):

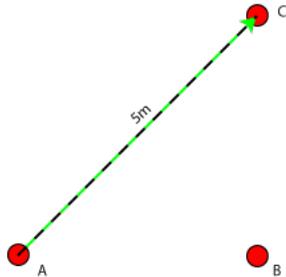


Figure 1

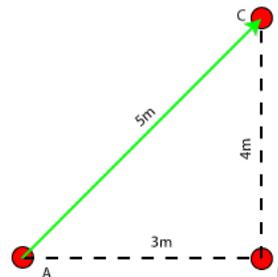


Figure 2

But be careful: displacement is NOT the same as **distance**. For example, if I walked from Point A for 3 meters to Point B and then I kept walking for 4 more meters to Point C, then the distance I moved would be $3\text{ m} + 4\text{ m} = 7\text{ m}$. However, the size of my *displacement* would still be 5 m (Figure 2).

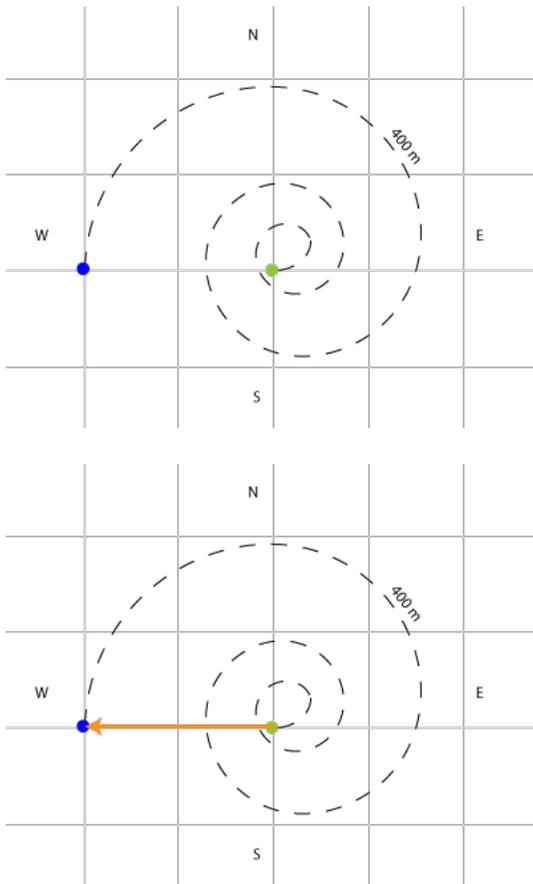
Think of it this way: if I draw a straight line from the place where I started moving directly to the place where I stopped moving, then the length of that line would be the size of my displacement. On the other hand, the distance I moved would be the length of the path that I actually walked. In both figures above, the displacement is drawn with a green arrow and the distance is drawn with a dotted black line.

The other difference between displacement and distance is that displacement is a vector quantity and distance is a scalar quantity. Assume up is North. Looking at Figure 1, we could say that my displacement was 5m [Northeast] and that my distance moved was 5m. The arrow for displacement denotes which direction I was traveling in – I started at Point A and went to Point C. If I had instead traveled from Point C to Point A, the arrow would point in the opposite direction and my displacement would be 5 m [Southwest]. The distance I traveled is still 5 m.

Notice:

1. the size of my displacement is only equal to distance if I travel in a straight line
2. displacement is represented as an arrow because it has a direction whereas distance is *not* represented by an arrow.

Example 1:



Frodo travels from home (green dot) to work as a jeweler (blue dot) everyday for 400m along the spiral shape shown. If every square is 10m long, what is his displacement? How far away is his home from his workplace?

We draw a straight arrow pointing from the starting point to the ending point to represent Frodo's displacement. Because of his starting and ending points, the size of his displacement also represents the distance between his home and workplace. Looking at the grid, we see that the displacement arrow falls along the length of two squares, each which measures 10m. In addition, the arrow points West. So Frodo's displacement is $10\text{m} + 10\text{m} = \mathbf{20\text{m}}$ [West] and the distance between his home and workplace is $\mathbf{20\text{m}}$. On the other hand, the distance that Frodo travels is 400m.

Velocity

We can describe how fast something is moving by stating its **velocity**. Like displacement, velocity is also a vector and so it has both a size and a direction. Therefore, if we are describing velocity, we must state both how fast something is going and which direction it is traveling. **Speed** is the scalar counterpart of velocity: it states only how fast something is going and not the direction.

The units of velocity and speed are a unit of length over a unit of time. For instance: m/s, km/h, and km/s are all examples of velocity and speed units. Velocity is displacement divided by time, whereas speed is *distance* divided by time.

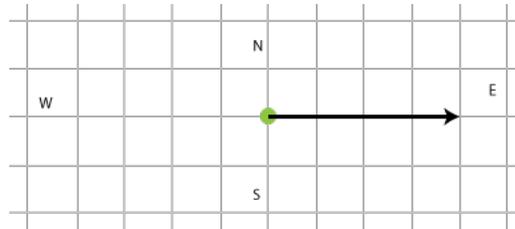
In other words, $v = \frac{d}{t}$ or $d = vt$. If v represents the **average velocity**, then d is the displacement. If instead v stands for the **average speed** of an object, d is the total distance. t is always time.

Note: average speed = size of average velocity only if distance = size of displacement.

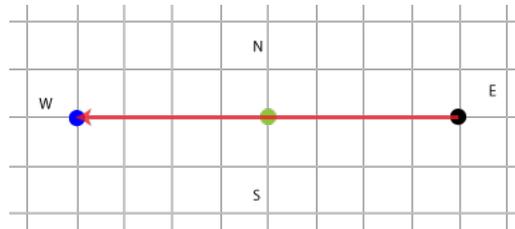
Example 1:

What is the average velocity of a car that moved 40 km East and 80 km West in 2 hours?
What is its average speed?

Let each square on the grid be 10 km long. Starting at the green dot, let's draw an arrow that spans 40 km East on our diagram:



Then, from our *new* starting point (indicated with a black dot), we move 80 km West. This is indicated by the red arrow:



Since velocity is displacement/time, we have to determine the displacement of the car. From our diagram, we see that the starting point (green) and the ending point (blue) are separated by 40 km [West]. So after the car has moved 40 km [East] and then 80 km [West], its resultant displacement is 40 km [West].

So the average velocity of the car is $\frac{40 \text{ km [West]}}{2 \text{ hrs}} = 20 \text{ km/hr [West]}$.

On the other hand, average speed is *distance* over time. The distance traveled by the car is 40 km + 80 km = 120 km. So the car's average speed is 120 km / 2 hrs = 60 km/hr

Acceleration

The term **acceleration** refers to how fast an object's velocity is changing. It is a vector quantity and has standard units $\text{m/s}^2 = \text{m/s/s}$ ("metres per second per second"). For example, if a boat's acceleration is 20 m/s^2 [East], then that boat's velocity is changing by 20 m/s [E] every second. If it starts at rest (velocity = 0 m/s) then after one second it will be moving 20 m/s [East]. After 2 seconds, it will be moving 40 m/s [East]. After 3 seconds, it will be moving 60 m/s [East], and so on. We will only be dealing with average acceleration.

We say that $v_1 = 0 \text{ m/s}$ is the boat's **initial velocity**. If the boat stops after 3 seconds and has acceleration 20 m/s^2 [E], then we say that $v_2 = 60$ [East] m/s is the boat's **final velocity**.

Example 1:

Frodo's eagle carries him from Tower 1 to Tower 2. If the eagle starts at rest and has acceleration 12 m/s^2 [Northeast], how fast is his eagle flying after 4 seconds?



<http://movie-screencaps.com/lord-rings-two-towers-2002/133/>

Since it starts at rest, the eagle's initial velocity is $v_1 = 0 \text{ m/s}$. Because its acceleration is 12 m/s^2 [Northeast], its velocity will increase by 12 m/s [Northeast] every second. So after one second, the eagle is going 12 m/s [Northeast]. After 2 seconds it is going an additional 12 m/s faster: so after 2 seconds it is going $12 + 12 = 2 \times 12 = 24 \text{ m/s}$ [Northeast]. After 3 seconds, it is going $3 \times 12 = 36 \text{ m/s}$ [Northeast]. After 4 seconds it is going $4 \times 12 = 48 \text{ m/s}$ [Northeast].

Example 2:

On the other hand, Gandalf's eagle starts with a velocity of 4 m/s [East] and lands 12 seconds later with a final velocity of 96 m/s [East]. How fast was Gandalf's eagle accelerating?

The change in the eagle's velocity is 96 m/s [East] - 4 m/s [East] = 92 m/s [East]. So over 12 seconds, the eagle's velocity changes by 92 m/s [East]. If we divide by the total time we can figure out how fast the velocity was changing *per second*:

$$\frac{92 \text{ m/s [East]}}{12 \text{ s}} = 23 \text{ m/s}^2 \text{ [East]}$$

The size of its velocity is changing by 23 m/s *per second*. Hence the units are $\text{m/s/s} = \text{m/s}^2$. So Gandalf's eagle is accelerating at 23 m/s^2 [East].

Exercises:

1. What is the acceleration of a car that travels in a straight line at a constant speed?

The acceleration is zero. There is no change in direction and no change in speed. Therefore, there is no acceleration.

2. A ball is dropped from the top of a building. After 2 seconds, its velocity is measured to be 19.6 m/s. Calculate its acceleration. Hint: “dropped” means it started from rest.

Acceleration is $19.6 \text{ m/s} \div 2 \text{ seconds} = 9.8 \text{ m/s}^2$ [down]

- * 3. You are traveling in a car that is moving at a velocity of 20 m/s [North]. Suddenly, a car 10 meters in front of you slams on its brakes. At that moment, you also slam on your brakes and slow to 5 m/s. Calculate the acceleration if it took 2 seconds to slow your car down. You can use negative numbers if you know them.

Hint: If you are slowing down, you are accelerating in the direction opposite to your motion.

$$\begin{aligned} \text{acceleration} &= \frac{\text{final velocity} - \text{initial velocity}}{\text{time passed}} \\ &= \frac{5 \text{ m/s [N]} - 20 \text{ m/s [N]}}{2 \text{ seconds}} \\ &= \frac{-15 \text{ m/s [N]}}{2\text{s}} \\ &= -7.5 \text{ m/s}^2 \text{ [N]} \text{ or } 7.5 \text{ m/s}^2 \text{ [S]} \end{aligned}$$

Notice above that I wrote acceleration using the direction [S]. This is because I am physically moving [N] and I am slowing down so my acceleration will be in the direction opposite to [N] (that is, my acceleration is in the direction [S]).

Dynamics

So far, we’ve analyzed the movement of objects without considering what causes those movements (this is called **kinematics**). When a pencil rolls off a table without being touched, what caused it to move? Why does it tend to *fall* without provocation? Why doesn’t it fly up instead? How come my stapler sits on top of the table and doesn’t fall through? In my opinion, questions like that are much more interesting than the question of how far the pencil fell or how fast it was moving. To figure out what made the pencil move, we will now delve into the study of **dynamics**, which focuses on the affects of force on the motion of physical objects. Kinematics and dynamics together make up mechanics (from the intro).

Mass & Force

We use the concept of **mass** to describe how much matter is inside a physical object. Anything that takes up space has matter in it. Generally, the bigger an object is, the more mass it has, but not always. If you could blow up a balloon to be the same size as an elephant, the balloon would still have less matter (or “stuff”) inside it than the elephant does. Although the balloon and the elephant are the same size, we say that the elephant has more mass.

We often weigh objects to figure out what their mass is. In the above example, the elephant would weigh more than the balloon, and that would be one indication that the elephant has more mass than the balloon. Having weight is a *result* of having mass. Weight and mass are not the same thing, though. Mass is a scalar quantity measured in kilograms (kg) and weight is a consequence of gravity acting on mass (see Activity #5).

A **force** is a push or a pull on an object. They can be balanced or unbalanced by other forces. Take this example: if you push on a box, then you are applying a force onto that box that is causing it to move. We say that the force of your push is unbalanced. If you and your friend pushed on the box from opposite directions and each of you is pushing just as hard as the other, the box would not move. We say that the force of your push is balanced out by the force of her push. If you pushed harder than your friend, then the box would move again in the direction that you were pushing. We say that the force of your push is unbalanced once again. **Gravity** is one example of a force. It is pulling you down toward the Earth.

Force is a vector quantity measured in Newtons (N), named for Sir Isaac Newton.

Newton’s Laws of Motion

Sir Isaac Newton was an English physicist and mathematician who is often credited with “discovering” gravity, but he did much more than that. In fact, Newton almost single-handedly invented the field of physics as we know it today. His three laws of motion are at the core of classical physics. In physics, a **law** is an undisputed truth that describes how the universe behaves under certain conditions.

Newton’s First Law

“An object at rest stays at rest and an object in motion stays in motion with the same speed and in the same direction unless acted upon by an unbalanced force.”

In simpler terms: if an object is sitting still, you must push it or pull it to make it move. If somebody starts pushing or pulling in the opposite direction as you with the same amount of strength, then the object will once again stop moving.

If instead the object is moving, it will be doing so with constant speed and in the same direction throughout its movement. If you want to do any of the following things to it, you must push it or pull it:

1. Make it go faster
2. Make it go slower
3. Make it change direction

By this logic, if you observe that an object is speeding up, slowing down, or changing direction, we say that an *unbalanced* force is acting on it and causing it to accelerate. Any object with zero acceleration will be moving with constant speed and direction (or not moving at all).

This law is also known as the Law of **Inertia**. Only objects with mass (stuff you can touch/stuff that takes up space) experience inertia.

Newton's Second Law

$$"F=ma"$$

Above, we saw that force is related to acceleration: you must apply an unbalanced force to an object with mass in order to make it move or to affect its movement in any way. This relationship is summarized in the equation above. F is the unbalanced force on the object with mass m . This mass will have acceleration a . If you are given any two of the three quantities force, mass, or acceleration, you can solve for the third quantity using this relationship. In this equation, force is measured in N, acceleration in m/s^2 and mass in kg.

Example 1:

You and your friend are still pushing that box from before. If you push to the right with a force of 40 N and they push to the left with a force of 20 N, what is the mass of the box? Assume that the box is accelerating at 2 m/s^2 [right].

The unbalanced force is going to be the sum of all the forces acting on the box. Using

Newton's second law, we get:

$$\begin{aligned}F &= ma \\40 \text{ N [right]} + 20 \text{ N [left]} &= m \times 2 \text{ m/s}^2 \text{ [right]} \\40 \text{ N [right]} - 20 \text{ N [right]} &= m \times 2 \text{ m/s}^2 \text{ [right]} \\20 \text{ N [right]} &= m \times 2 \text{ m/s}^2 \text{ [right]} \\20/2 &= m \\10 &= m\end{aligned}$$

Since mass is measured in kg, we say that the mass of the box is 10 kg.

I chose right to be the “positive direction” because I knew that the box was accelerating to the right and so this would be the direction of the unbalanced force. Therefore, vectors pointing left are pointing in the negative right direction.

If your friend starts pushing at 40 N [left] your force will once again be balanced and you would plug 0 in for F : there is no unbalanced force. Whenever there is no unbalanced force, the acceleration is also 0. Whenever there *is* an unbalanced force, the acceleration is in the same direction. In this example, the unbalanced force was to the right so acceleration was also to the right.

Newton's Third Law

“For every action, there is an equal and opposite reaction.”

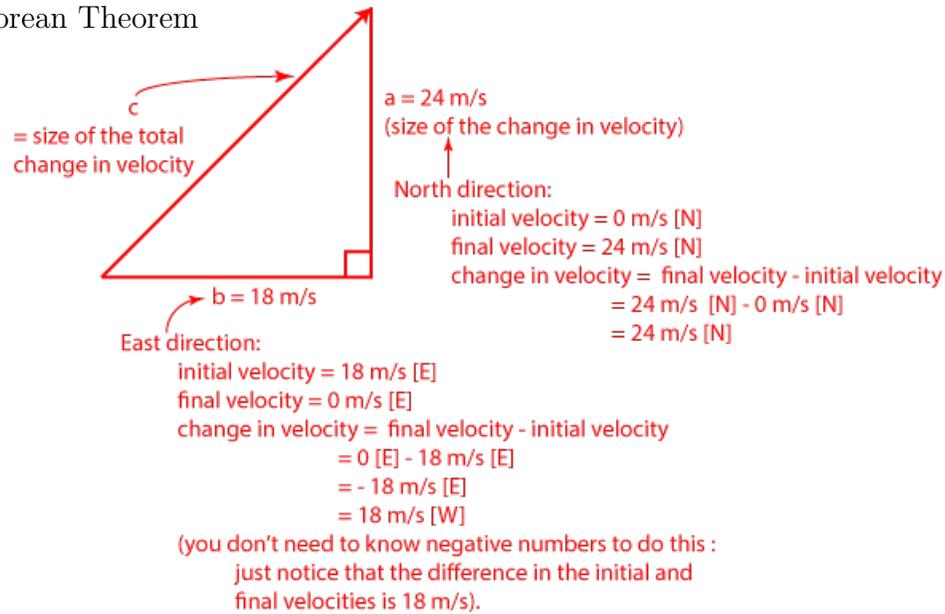
Any force applied by object A onto object B results in a force of *equal size* being applied by object B onto object A in the opposite direction. In other words, if you push the wall with a force of 40 Newtons [right], the wall pushes back on you with a force of 40 Newtons [left]!

Activities

1. If I am sailing on the Pacific ocean with an initial velocity of 18 m/s [East] and I gradually speed up for 30 seconds until my final constant velocity is 24 m/s [North], what was my average acceleration during those 30 seconds?

Hint: Average acceleration is the change in average velocity over time.

Hint Hint: Because East and North are not **collinear**, you will have to use the Pythagorean Theorem



I use the Pythagorean Theorem to solve for side c :

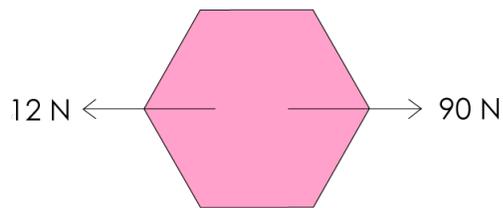
$$\begin{aligned}c^2 &= a^2 + b^2 \\c^2 &= 24^2 + 18^2 \\c^2 &= 576 + 324 = 900 \\ \Rightarrow \sqrt{c^2} &= \sqrt{900} \\c &= \sqrt{900} = 30\end{aligned}$$

Since everything is in the same units, side c will be 30 m/s “long”. So the total change in average velocity is 30 m/s [NE] (since I sail East then end up going North, my total change in velocity must be in the direction Northeast as you can see from the diagram). Now, it took me 30 seconds to change speeds so my acceleration will be the change in

total average velocity over time:

$$\begin{aligned}\text{acceleration} &= \frac{30 \text{ m/s [NE]}}{30 \text{ sec}} \\ &= 1 \frac{\text{m}}{\text{s}^2} \text{ [NE]}\end{aligned}$$

2. Describe a situation in which you can accelerate even though your speed doesn't change. This would be the case if I was going in a circle without changing speed. Since my direction would constantly be changing, so would my acceleration and velocity which are vector quantities.
3. Consider the following "Free Body Diagram":



- (a) Find the unbalanced force (also known as the "net force").

$$\begin{aligned}\text{net force} &= 90 \text{ N [right]} + 12 \text{ N [left]} \\ &= 90 \text{ N [right]} - 12 \text{ N [right]} \\ &= 78 \text{ N [right]}\end{aligned}$$

- (b) If in the diagram above, the hexagon had a mass of 5 kg, how much would it be accelerating and in what direction would it be accelerating?

$$\begin{aligned}F &= ma \\ 78 \text{ N [right]} &= (5 \text{ kg}) \times a \\ a &= 78 \text{ N [right]} \div 5 \text{ kg} \\ a &= 15.6 \text{ m/s}^2 \text{ [right]}\end{aligned}$$

- (c) With how much force would you have to push to make the hexagonal object accelerate at 10 m/s^2 ?

Assume that you are pushing the box right. The unbalanced force must also be in the direction [right]. You could have chosen any direction, but you must choose a direction because force and acceleration are vector quantities.

$$F = ma$$

$$F = (5 \text{ kg}) \times (10 \text{ m/s}^2 \text{ [right]})$$

$$F = 50 \text{ N [right]}$$

- (d) If I had a new shiny blue hexagon that accelerated at 10 m/s^2 when the unbalanced force in part (a) was applied to it, how much mass would my shiny new hexagon have?

The unbalanced force in part (a) is to the [right] so the acceleration mentioned in this question is also to the [right].

$$F = ma$$

$$78 \text{ N [right]} = m \times 10 \text{ m/s}^2 \text{ [right]}$$

$$m = 78 \text{ N [right]} \div 10 \text{ m/s}^2 \text{ [right]}$$

$$m = 7.8 \text{ kg}$$

4. If Box A has twice the mass of Box B and you push both boxes with the same amount of force, which box will have the greater acceleration?

If Box B has mass m then Box A has mass $2m$. Let's look at Each case separately. First, Box A:

$$F = 2m \times a$$

$$a = \frac{F}{2m}$$

$$a = \frac{1}{2} \times \frac{F}{m}$$

Next, Box B:

$$F = m \times a$$

$$a = \frac{F}{m}$$

Since $\frac{1}{2} \times \frac{F}{m} < \frac{F}{m}$, Box B (the box with lower mass) will accelerate more when the same Force is applied to Box A and Box B. Intuitively, you know this is true: it's easier to move a smaller object than it is to move a bigger object.

5. Having weight is a result of gravity acting on mass. We often hear the phrase “the Earth is weighing me down”, which is literally true because weight is just the force of gravity. When you feel yourself being pulled toward the Earth by gravity, you are

feeling your weight. We can measure any object's weight by finding the force of gravity that acts on them via Newton's second law $F = ma$. On Earth, we say that you are accelerating toward the ground as a result of gravity (you are being pulled toward the ground). Acceleration due to gravity on Earth is 9.8 m/s^2 [down]. So an object with mass 50 kg would weigh $F = (50) \times (9.8) = 490 \text{ N}$ on Earth. Notice that because weight is the *force* of gravity acting on an object, it is measured in Newtons (N).

- (a) How much would an object of mass 42 kg weigh on Earth?

$$\begin{aligned} \text{mass} \times 9.8 \text{ m/s}^2 \text{ [down]} &= \text{weight} \\ 42 \text{ kg} \times 9.8 \text{ m/s}^2 \text{ [down]} &= 411.6 \text{ N [down]} \end{aligned}$$

When we talk about how much something weighs, we normally just talk about the *size* of the force of gravity because we know that gravity always pulls down. I will exclude direction when I talk about weight for simplicity. So the object weighs 411.6 N .

- (b) The average African bush elephant weighs $53,900 \text{ N}$ on Earth. How much mass does it have?

$$\begin{aligned} \text{mass} \times 9.8 \text{ m/s}^2 &= \text{weight} \\ \text{mass} &= \text{weight} / 9.8 \text{ m/s}^2 \\ \text{mass} &= 53,900 \text{ N} / 9.8 \text{ m/s}^2 \\ \text{mass} &= 5500 \text{ kg (wow!)} \end{aligned}$$

- (c) The acceleration due to gravity on the Moon is one sixth of what it is on Earth. How much would an average 12 year old (42 kg) and an average African bush elephant weigh on the moon?

Acceleration due to gravity on the moon $= 9.8 \div 6 \text{ m/s}^2 = 1.63333\dots \text{ m/s}^2$. You would weigh one sixth of what you do on Earth. Generally:

$$\begin{aligned} \text{weight on moon} &= \text{mass} \times (9.8 \div 6 \text{ m/s}^2) \\ \text{weight on moon} &= \text{weight on Earth} \div 6 \end{aligned}$$

For the average 12 year old:

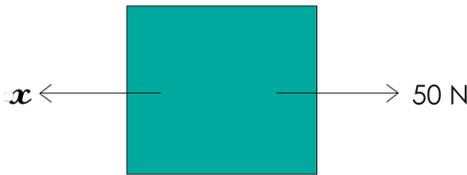
$$\begin{aligned} \text{weight on moon} &= \text{weight on Earth} \div 6 \\ &= 411.6 \text{ N} \div 6 \\ &= 68.6 \text{ N} \end{aligned}$$

For the average African bush elephant:

$$\begin{aligned}\text{weight on moon} &= 53,900 \text{ N} \div 6 \\ &= 8983.3 \text{ N}\end{aligned}$$

Interesting note: On a scale, you see pounds (lbs) and kilograms (kg). Pounds (used in America) are a unit of weight (like Newtons are) but kilograms (used in Canada) are a unit of mass! Tsk, tsk...

- * 6. Solve for x in Newtons if the size of acceleration of the block is 10 m/s^2 :



What is the new x (call it x') if I add another block of equal mass on top? (Assume the same acceleration and forward force).

Hint: both your answers should have an m in them.

What is x if $m = 5 \text{ kg}$? What is x' ?

The unbalanced force is $50 \text{ N [right]} - x$ (since the forces are in the opposite directions, we make sure that every direction faces right. A -5 N force left is the same as a $+5 \text{ N}$ force right). For a **single block** with mass m , we plug this into Newton's second law, remembering that dividing the force unit N by the acceleration unit m/s^2 results in the mass unit kg :

$$\begin{aligned}F &= ma \\ 50 \text{ N [right]} - x &= m \times (10 \text{ m/s}^2) \\ -x &= 10m - 50 \text{ N} \\ x &= (50 - 10m) \text{ N}\end{aligned}$$

If I add another box of equal mass on top, then the new mass of the whole system is $m + m = 2m$. Therefore:

$$x' = (50 - (10 \times 2m)) \text{ N} = (50 - 20m) \text{ N}$$

If $m = 5 \text{ kg}$, Then:

$$\begin{aligned}
 x &= (50 - 10m) \text{ N} \\
 &= (50 - 10(5)) \text{ N} \\
 &= (50 - 50) \text{ N} \\
 &= 0 \text{ N}
 \end{aligned}$$

If the box had mass 5 kg and was accelerating at 10 m/s^2 , the only force acting on it would be a force of 50 N [right]. This means it would also be accelerating to the right:

$$a = 10 \text{ m/s}^2 \text{ [right]}$$

On the other hand, if we stack two boxes on top of each other:

$$\begin{aligned}
 x' &= (50 - 20m) \text{ N} \\
 &= (50 - (20 \times 5)) \text{ N} \\
 &= (50 - 100) \text{ N} \\
 &= -50 \text{ N}
 \end{aligned}$$

So with two boxes, the net force becomes $50 \text{ N [right]} - (-50 \text{ N}) = 100 \text{ N [right]}$. You would have to reverse the direction of the force x so that it was facing the same way as the original force of 50 N. The force x would also have to be 50 N [right] and therefore the total force on the system has to be 100 N [right] in order to get 2 boxes of mass 5 kg each to move with an acceleration of 10 m/s^2 [right].

* denotes a hard question.