



Intermediate Math Circles

Wednesday November 2 2016

Mathematical Games II

Binary Numbers

Consider the number 367. This is notation for $3 \times 100 + 6 \times 10 + 7$. If you use the notation of exponents, you can write this as $3 \times 10^2 + 6 \times 10^1 + 7 \times 10^0$. Usually, we write numbers in *base 10*, like this. The base is 10, which is one more than our largest digit, 9. This is because once you have ten of a power of 10, it is a larger power of 10. For example, ten 10s is 100 and ten 100s is 1000.

There are times when it is more convenient to write numbers in another base. Base 2 is particularly popular, meaning we write the number as a sum of powers of 2. That is, 2, 4, 8, 16, 32, 64, 128, Just as in the base 10 case, the largest allowed digit is one smaller than the base, which, in the case of base 2, is 1. This is why binary is nice. There are only two digits: 0 and 1. Here is how you write 50 in base 2 or binary:

- Find the largest power of 2 less than or equal to 50. It is $32 = 2^5$. Now write $50 = 32 + 18$.
- The leftover part is 18. Find the largest power of 2 which is less than or equal to 18. It is 16, so now we write $50 = 32 + 16 + 2$
- 2 is a power of 2, so now we stop, having written $50 = 2^5 + 2^4 + 2^1$.
- Fill in the “placeholders” (just like base 10) to get $50 = 1 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$.
- Extract the digits to get the binary expansion for 50 which is 110010.

It is a fact that every integer has a binary expansion, and that there is only one. Let's do 99.

- The largest power of 2 which is less than or equal to 99 is 64, so we write $99 = 64 + 35$.
- Continuing, the largest power of 2 which is less than or equal to 35 is 32, so $99 = 64 + 32 + 3$.
- Now split 3 into 2+1 to get $99 = 64 + 32 + 2 + 1 = 2^6 + 2^5 + 2^1 + 2^0$.
- The binary expansion of 99 is 1100011. Note that a 0 had to go wherever a power of two was “missing”.

We often write the base as a subscript at the end of the number. For example, 11_2 means 11 interpreted as a binary number, which is three. Also, note that the binary expansion of a power of 2 is just 1 followed by the same number of 0s as the power. For example, 2^5 is 100000_2 . Binary expansions are at the core of the general strategy for Nim. Here it is

1. Write all of the pile sizes in binary and write them in a right-justified column.
2. Count the number of 1s in each column. If it is even in every column, it is a previous player win.
3. Otherwise, it is a next player win. The correct move is to remove stones from one pile so that every column will have an even number of 1s.



For example, let's examine $2 \oplus 5 \oplus 4 \oplus 6$. The pile sizes written in binary are 10, 101, 100, 110. Written in a right-justified column gives

$$\begin{array}{r} 1\ 0 \\ 1\ 0\ 1 \\ 1\ 0\ 0 \\ 1\ 1\ 0 \end{array}$$

There are three 1s in the leftmost column, and 1 in the right most, so this is a next player win. A correct move is any move that leaves an even number of 1s in each column. If the bottom row were 11 instead of 110, every column would have an even number of 1s. Let's remove enough stones from the pile of 6 to make that happen. The pile currently has 6 stones, and we want it to have 11_2 which is 3. Therefore, a move that will lead to a win is to remove 3 stones from that pile to leave 3. The state $2 \oplus 5 \oplus 4 \oplus 3$ in right-justified array form is

$$\begin{array}{r} 1\ 0 \\ 1\ 0\ 1 \\ 1\ 0\ 0 \\ 0\ 1\ 1 \end{array}$$

which is a previous player win since every column has an even number of 1s.

Zackendorf decompositions

Zackendorf decompositions are, in a way, like binary expansions. Recall that the Fibonacci sequence is 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, Each term is obtained by taking the sum of the previous two terms. Using a similar method to writing binary expansions, we can write a sort of "Fibonacci expansion", known as a Zackendorf expansion or Zackendorf decomposition. Here is how you find the Zackendorf decomposition of 86.

- Find the largest Fibonacci number which is less than or equal to 86. It is 55, so write $86 = 55 + 31$.
- Find the largest Fibonacci number which is less than or equal to 31. It is 21, so write $86 = 55 + 21 + 10$.
- Continue this way to write $86 = 55 + 21 + 8 + 2$. 2 is a Fibonacci number, so the process stops here. This sum is the Zackendorf decomposition.

Kayles

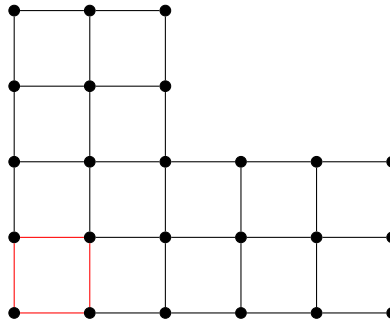
In Kayles, there are pins (think bowling pins) arranged in a circle. A legal move is to remove any one pin, or remove two adjacent pins. Adjacent means that they were next to each-other in the beginning. That is, two pins do not become adjacent if the pin between them is removed. Two players alternate removing pins according to these rules, and the winner is the player that takes the last pin.

Chomp

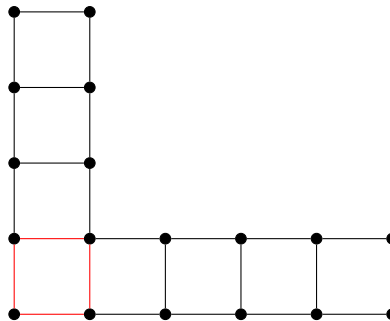
Chomp is played on a partial rectangular grid (or a chocolate bar) with the bottom left cell identified as bad (or poisoned, in the case of chocolate). Players take turns "chomping" entire



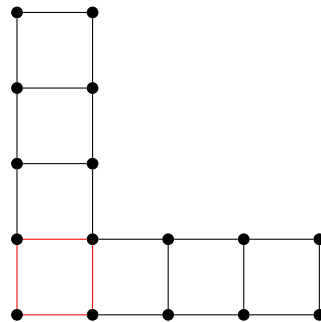
north-east sections from the grid. The player that is forced to take the identified cell loses. For example, the game could start like this:



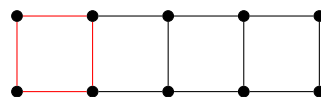
and the first player might (unwisely) chomp the whole inner layer leaving



Then the second player might remove a single cell from the bottom row to leave



Then the first player might chomp the left column to leave



at which point the second player will take all but the bad cell, and the first player will lose.