Intermediate Math Circles
Wednesday November 23 2016
Problem Set 8

1. The point of this first exercise is to get warmed up and recall what we learned last time.

   (a) Find the remainder when 3434 is divided by 17.
   Solution

$$\begin{aligned} 3434 &\equiv 3400 + 34 \pmod{17} \\ &\equiv 0 + 0 \pmod{17} \\ &\equiv 0 \pmod{17} \end{aligned}$$

   Therefore the remainder is 0 which means that 17 divides 3434.

   (b) What is the tens digit of $(99)^{881}$?
   Solution

$$\begin{aligned} (99)^{881} &\equiv (-1)^{881} \pmod{100} \\ &\equiv -1 \pmod{100} \\ &\equiv 99 \pmod{100} \end{aligned}$$

   Therefore the tens digit is 9.

   (c) January 1, 2017 is on a Sunday. What day of the week will February 14, 2018 be on?
   Solution
   Recall from last time that $365 \equiv 1 \pmod{7}$ so that January 1, 2018 will be a Monday. Then, $30 + 14 \equiv 2 \pmod{7}$ so that February 14, 2018 will be a Wednesday.

2. Calculate the remainder when $(17)^{125}$ is divided by 5. (Hint: Use Fermat's Little Theorem)
   Solution

$$\begin{aligned} 17^{125} &\equiv (2)^{125} \pmod{5} \\ &\equiv 2^{5\times5\times5} \pmod{5} \\ &\equiv (2^5)^{5\times5} \pmod{5} \\ &\equiv 2^{5\times5} \pmod{5} \text{ by FLT} \\ &\equiv (2^5)^5 \pmod{5} \\ &\equiv 2^5 \pmod{5} \text{ by FLT} \\ &\equiv 2 \pmod{5} \text{ by FLT.} \end{aligned}$$

3. True of False: Every number of the form $a^2 - a$ is even.
   Solution
   True. Fermat's little theorem with $p = 2$ says that $a^2 \equiv a \pmod{2}$ so that $a^2 - a$ is divisible by 2 for any $a \in \mathbb{Z}$. Therefore $a^2 - a$ is always even.

4. What is $15002^{(3^4)}$ congruent to modulo 3? (i.e. 0, 1, or 2?)

   Solution

   Using a similar argument as in problem (2) we see that

$$
\begin{aligned}
15002^{3^4} &\equiv (2)^{3^4} \pmod 3 \\
&\equiv 2^{3\times3\times3\times3} \pmod 3 \\
&\equiv (2^3)^{3\times3\times3} \pmod 3 \\
&\equiv 2^{3\times3\times3} \pmod 3 \\
&\equiv (2^3)^{3\times3} \pmod 3 \\
&\equiv (2^3)^3 \pmod 3 \\
&\equiv 2^3 \pmod 3 \\
&\equiv 2 \pmod 3.
\end{aligned}
$$

5. Show that the check digit *always* detects an error made when making exactly one typo in a UPC.

   Solution

   Suppose that a (correct) UPC for an item is $(a_1, \ldots, a_{11}, a_{12})$. Now suppose that $(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_{12})$ is accidentally entered but the check digit a_{12} remains the same. We show that this can only happen when $a_i = b_i$. That is, when no mistake was made! Well, depending on if i is even or odd, this means that $a_i \equiv b_i \pmod{10}$ or $3a_i \equiv 3b_i \pmod{10}$. In the latter case, we see by multiplying both sides of the congruence by 7 that this implies $a_i \equiv b_i \pmod{10}$. Therefore, in any case, we have that $a_i \equiv b_i \pmod{10}$ and so $a_i - b_i \equiv 0 \pmod{10}$. Hence $a_i - b_i$ is divisible by 10. However, $a_i - b_i$ is too small to be divisible by 10 unless $a_i = b_i$. Therefore $a_i = b_i$ and so the only time the check digit remains constant when up to one mistake has been made is when no mistake has been made at all!

6. Which of the following UPC's have correct check digits?

   0 30955 61982 8

   0 98756 10001 3

   7 90524 16397 7

   Solution All of them are correct!

7. What is the check digit of the UPC $(0,3,0,9,5,5,1,6,9,8,2,\star)$? What do you notice about this answer when compared to the above question?

   Solution

   Let x be the check digit of the above UPC. Then, we see that

   $$(0,3,0,9,5,5,1,6,9,8,2,x)\cdot(3,1,3,1,3,1,3,1,3,1,3,1)\equiv 0 \pmod{10}.$$

   Therefore $3+9+15+5+3+6+27+8+6+x\equiv 0 \pmod{10}$ and so $2+x\equiv 0 \pmod{10}$. Since we know that $0\le x\le 9$ we must have that $x=8$. Notice that compared to the left-most UPC pictured above, a switch of adjacent digits was made but the check digit did not detect an error!

8. Determine when the check digit of a UPC does *not* detect an error made by switching two adjacent numbers

   Solution

   Suppose that the correct UPC for an item is $(a_1,a_2,\ldots,a_{12})$ but a mistake is made and the identification number is entered as

   $$(a_1,a_2,\ldots,a_{i+1},a_i,\ldots,a_{12}),$$

   where $i\le 10$. That is, the check digits remain the same even though a mistake is made involving the switch of two adjacent digits. Of course, we may assume that $a_i\ne a_{i+1}$ for otherwise no mistake was actually made - lucky!. Then we see that

   $$(a_1,\ldots,a_{12})\cdot(3,1,\ldots,3,1)\equiv(a_1,a_2,\ldots,a_{i+1},a_i,\ldots,a_{12})\cdot(3,1,\ldots,3,1) \pmod{10}.$$

   It then follows that $3a_i+a_{i+1}\equiv a_i+3a_{i+1} \pmod{10}$ so that $2a_i\equiv 2a_{i+1} \pmod{10}$. Therefore $2(a_i-a_{i+1})$ is divisible by 10. It is then easily verified that this happens exactly when $|a_i-a_{i+1}|=5$. Therefore, this method does not detect an error made by switching two adjacent digits when the digits have a difference of plus or minus 5.

9. In Florida, the fourth and fifth digits from the end of a driver's license number give the year of birth. The last three digits for a male with birth month m and birth date b are represented by $40(m-1)+b$. For females the digits are $40(m-1)+b+500$. Determine the dates of birth of people who have last five digits 42218 and 53953.

   Solution

   First let us consider the license number 42218. Easily, the first two digits 42 tell us that this individual was born in 1942. Now, we consider the last three digits 218. As $218<500$ we know that this individual is a male. Therefore $218=40(m-1)+b$. As $218\equiv 18 \pmod{40}$ we have that $b=18$. Therefore $200=40(m-1)$ so that $m-1=5$ and $m=6$. We conclude that this male was born on June 18, 1942.

   No consider the license number 53953. This person was born in the year 1953. Moreover, as $953>500$ we know that this individual is a female. Since $953-500=453$ and $453\equiv 13 \pmod{40}$ we have that $b=13$. Moreover, $453-13=40(m-1)$ so that some quick algebra tells us that $m=12$. Therefore this female was born on December 13, 1953.

10. For driver's license numbers issued in New York prior to September of 1992, the three digits preceding the last two of the number of a male with birth month m and birthdate b are represented by $63m + 2b$. For females the digits are $63m + 2b + 1$. Determine the dates of birth and genders which correspond to the numbers 248 and 601.

    Solution

    First consider an individual with licence number (well, the three digits from the end) 248. Since $63 \times 3 = 189$ we see that $248 \equiv 59 \pmod{63}$. Now, 59 either represents $2b$ or $2b+1$ depending on the gender of the individual. Noting that $2b$ is always even and $2b + 1$ is always odd it must be the case that the individual is female and that $2b+1 = 59$. It then follows that $b = 29$. Moreover, $248 = 63m + 59$ and so $m = \frac{189}{63} = 3$. Therefore this female was born on March 29.

    Now consider an individual with license number 601. We begin by noting that $601 \equiv 601 - 630 \equiv -29 \equiv 34 \pmod{63}$. Therefore, since 34 is even, $2b = 34$ and $b = 17$. Moreover, this individual must be male. Continuing with our calculation, $601 = 63m + 34$ and so $m = \frac{567}{63} = 9$. Therefore this male was born on September 17.

11. The state of Utah appends a ninth digit a_9 to an eight-digit driver's license number $a_1a_2 \cdots a_8$ so that $(a_1, a_2, \ldots, a_9) \cdot (9, 8, 7, 6, 5, 4, 3, 2, 1) \equiv 0 \pmod{10}$. If you know that the license number 149105267 has exactly one digit incorrect, explain why the error cannot be in position 2,4,6, or 8.

    Solution

    If exactly one error has been made then this implies that the correct license number is $(1, x, 9, 1, 0, 5, 2, 6, 7)$, $(1, 4, 9, x, 0, 5, 2, 6, 7)$, $(1, 4, 9, 1, 0, x, 2, 6, 7)$, or $(1, 4, 9, 1, 0, 5, 2, x, 7)$ for some $x \in \{0, 1, 2, \ldots, 9\}$. Considering each of these licence numbers in the equation $(a_1, a_2, \ldots, a_9) \cdot (9, 8, 7, 6, 5, 4, 3, 2, 1) \equiv 0 \pmod{10}$, it is seen that either

    $$8x \equiv 7 \pmod{10},$$

    $$6x \equiv 1 \pmod{10},$$

    $$4x \equiv 5 \pmod{10},$$

    or

    $$2x \equiv 7 \pmod{10},$$

    for some $0 \leq x \leq 9$. A quick check shows that none of the above equations has a solution! Therefore the mistake could not have been in positions 2,4,6, or 8.