Intermediate Math Circles
Wednesday November 2 2016
Problem Set 5

Here are the November 2 exercises. (6), (7), (8), and (9) are from October 26.

1. Find the binary expansions of 3, 7, 10, 15, 21, 30, 32, and 53.

Solution

base 10	base 2
3	11
7	111
10	1010
15	1111
21	10101
30	11110
32	100000
53	110101

2. Find the binary expansions of 6, 14, 20, 30, 42, 60, 64, and 106. Compare your answers to the answers in the previous problem. Explain your observation. [Hint: You don't need to do all of them to notice the pattern. Start with the small ones.]

Solution

The numbers in this list are twice the numbers in the list from exercise 1. Let's make a table that compares the two:

n in decimal	n in binary	$2n$ in decimal	$2n$ in binary
3	11	6	110
7	111	14	1110
10	1010	20	10100
15	1111	30	11110
21	10101	42	101010
30	11110	60	111100
32	100000	64	1000000
53	110101	106	1101010

You should notice that the expansions in the last column are simply the expansions in the second column with a zero on the end. Just as multiplying by 10 in base 10 corresponds to appending a 0 to the right of the number, multiplying by 2 in base 2 corresponds to appending a 0 to the right of the number.

3. Pick two 2-digit numbers and write them in binary. Add and multiply them in binary using the regular method for addition and multiplication. Keep in mind that $1 + 1 = 10$ and $1 + 1 + 1 = 11$ in binary. Convert the answers back to decimal and check to see if you were correct. Try another pair of numbers if you like! You don't have to stick to two digit numbers.

   Solution

   I will choose 20 and 39. Their binary expansions are 10100 and 100111. Let's right-justify them and perform the addition algorithm:

$$\begin{array}{cccccc} & & {\scriptstyle 1} & & & \\ 1 & 0 & 0 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 1 & 1 \end{array}$$

   Multiplication is really easy:

$$\begin{array}{cccccccccc} & & & & 1 & 0 & 0 & 1 & 1 & 1 \\ & & & & & 1 & 0 & 1 & 0 & 0 \\ \cline{5-10} & & & {\scriptstyle 1} & 0^{1} & 0 & 0 & 0 & 0 & 0 \\ & & {\scriptstyle 1} & 0 & 0 & 0 & 0 & 0 & 0 & \\ & {\scriptstyle 1} & 1 & 0 & 0 & 1 & 1 & 1 & & \\ & 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 1 & 0 & 0 & 1 & 1 & 1 & & & & \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}$$

   If you convert the results back to decimal, you get that 111011_2 is 59, and 1100001100_2 is 780. Of course, $20 + 39 = 59$ and $20 \times 39 = 780$, so this works. In fact, it will always work.

4. Recall that a number is divisible by 3 exactly when the sum of its digits is divisible by 3. For example, since $2 + 7 + 9 + 1 + 5 = 24$, which is a multiple of 3, the number 27915 is a multiple of 3.

   (a) Find an example to show that this does not work for binary expansions.

   (b) Using inspiration from the "divisibility by 11" trick, find a trick that will tell you whether or not a number, written in binary, is divisible by 3.

   Solution

   A simple example is that $3 = 11_2$, and $1 + 1 = 2$ which is not a multiple of 3. Another example is $30 = 11110_2$, and $1 + 1 + 1 + 1 = 4$, which is also not a multiple of 3. However, if we take *alternating sum*, it does work. In the case of 3, the alternating sum of its binary expansion is $1 - 1 = 0$, and in the case of 30, the alternating sum of its binary expansion is $1 - 1 + 1 - 1 + 0 = 0$, which is a multiple of 3. For a more exciting example, 327, which is 3×109 has a binary expansion of 101000111, and its alternating sum is $1 - 0 + 1 - 0 + 0 - 0 + 1 - 1 + 1 = 3$, which is a multiple of 3.

5. For each Nim state, determine whether or not it is a next or previous player win. For those that are next player win, give a correct next move. [Hint: Use the first exercise.]

   (a) $53 \oplus 32 \oplus 30 \oplus 15$

   (b) $15 \oplus 7 \oplus 3$

   (c) $30 \oplus 21 \oplus 10$

   Solution

   In question 1, we wrote all of the relevant numbers in binary. Here are the tables, with the number of 1s in each column at the bottom:

   (a)
   $$\begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 0 \\ & & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 3 & 2 & 2 \end{array}$$

   (b)
   $$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ \hline 1 & 2 & 3 & 3 \end{array}$$

   (c)
   $$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ & 1 & 0 & 1 & 0 \\ \hline 2 & 2 & 2 & 2 & 1 \end{array}$$

   We can see that there is at least one column in each case with an odd number of 1s, so they are all next player wins. For (a), if we could turn the third row into 11010, we would be left with

   $$\begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 0 \\ & & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 & 2 \end{array}$$

   which has an even number of 1s in each row. Therefore, we want to turn the pile of 30 into a pile of 11010_2, which is 26, so a correct move is to remove 4 stones from the pile of 30. In (b), we want to turn the first row into 100, which means changing it from having 15 (1111 in binary) stones to having 4 (100 in binary) stones. Therefore, a correct move is to remove 11 stones from the pile of 15. In (c), the only odd sum occurs in the right-most column, so we can just remove 1 from the pile corresponding to the middle row, which is the pile of 21.

6. Find the Zackendorf decompositions of 10, 15, 20, 500, 610, and 1000.

   Solution

10	$2 + 8$
15	$2 + 13$
20	$2 + 5 + 13$
500	$34 + 89 + 377$
610	610
1000	$13 + 987$

7. Play Fibonacci Nim for a few minutes. Try to find a winning strategy for a few small examples. Can you find a winning strategy in general? Warning: This one is hard. The general strategy involves the Zackendorf decomposition of the number.

   Solution

   The general strategy is actually quite simple. If the pile starts with a number which is **not** in the Fibonacci sequence, Player 1 has a winning strategy. Otherwise, Player 2 does. The winning move is to always, if possible, take the smallest number in the Zackendorf decomposition of the number of stones. It should work out that the Player without the winning strategy will never be legally allowed to take enough stones to achieve this. The strategy is quite easy, but the proof that it works is tedious.

8. Explain why no two numbers in a Zackendorf decomposition are consecutive Fibonacci numbers. Here consecutive means they appear next to one another in the Fibonacci sequence.

   Solution

   The key is that the sum of two consecutive Fibonacci numbers is a Fibonacci number. Remember, when computing the Zackendorf decomposition of a number, you always use the largest Fibonacci number that you can. If you ended up using two consecutive Fibonacci numbers, you could have used a larger one (their sum) earlier.

9. Explain why the same number can not occur twice in the Zackendorf decomposition of a number.

   Solution

   This one is similar to the previous one. Twice a Fibonacci number is bigger than the next Fibonacci number, except in the case of 1 and 2 when they are equal. For example, $3 + 3 = 6 > 5$ and $13 + 13 = 26 > 21$. Therefore, if the same number occurred twice, you could have chosen a larger one earlier when constructing the Zackendorf decomposition.

10. Find a strategy for Kayles.

    Solution

    In this game, Player 2 always has a winning strategy, unless there are two pins. The game starting with 2 pins is kind of silly. It doesn't even really mean anything for two pins to be arranged in a circle.

    After Player 1 moves, the circle is broken and you can imagine that the pins are in a line. Player 2 should remove either 1 pin or two pins from the middle of the line so that there are two equal lines. Then Player 2 can just copy Player 1's moves on the opposite side. For example, if the game starts with a circle of 20 pins and Player 1 removes 1, you can now imagine the game as a line of 19 pins. Player 2 should then remove the 10th pin, leaving two lines of 9. After that, if Player 2 always copies Player 1's move, they will eventually win!

11. Show that Player 1 has a winning strategy in Chomp in the case that the game starts with a rectangular grid. [Hint: Nobody knows a winning strategy, but you can show that there is one. If Player 2 had a winning strategy, how would they respond to Player 1 removing just the top right cell?]

    Solution

    This one is strange. First, remember that this is a combinatorial game, so exactly one of the players has a winning strategy. We will show that it can not be Player 2, so it has to be Player 1. Let's imagine that Player 2 has a winning strategy. This means no matter what Player 1 does, Player 2 has a move that will leave Player 1 with a previous player win. This includes the situation where Player 1, on their first move, removes the top right cell only. No matter how Player 2 responds, they leave the grid in a configuration that could have been reached by Player 1 on their first move. You should stop reading now and convince yourself of this. It is essentially because any legal first move for Player 1 removes at least the top right cell. Since we are assuming Player 2 has a winning strategy, Player 2 has a move that leaves Player 1 with a previous player win. From what we said earlier, this means Player 1 could have left Player 2 with a previous player win, which means Player 1 has a winning strategy.

    To conclude, we have shown that if Player 2 has a winning strategy, then Player 1 has a winning strategy. This is nonsense, so Player 2 can not possible have a winning strategy. Therefore, Player 1 has a winning strategy.

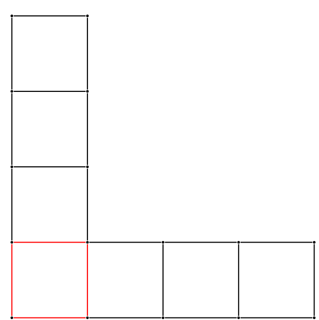
12. No general winning strategy for Player 1 is known for a rectangular grid. However, you can find one for some special cases. Find a winning strategy for the Player 1 in the following situations:

    (a) A square grid.

    (b) A $2 \times n$ grid.

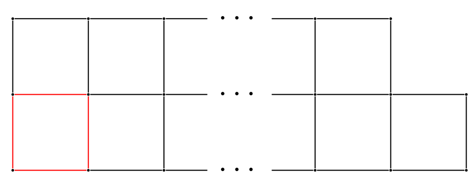
    Solution

    (a) For a rectangular grid, Player 1 should take all but the left column and the top row, leaving something that looks like

    Now Player 1 can use the same strategy as they do in Nim when there are two equal piles. That is, mirror what Player 2 does: If Player 2 takes 2 from the left column, Player 1 takes 2 from the bottom row. If Player 2 takes 12 from the bottom row, Player 1 takes 12 from the left column, and so on. This will eventually end up with Player 2 stuck with the poisonous cell.

(b) The plan for Player 1 should be to always leave Player 2 with the top row one cell shorter than the bottom row. That is, the grid should look like this:



On their first move, they should take the top right cell only. No matter what Player 2 does, Player 1 always has a move to leave the grid this way. Since players must take some cells on every turn, the grid has to shrink, so eventually Player 1 will leave Player 2 with no cells in the top row and one cell in the bottom row. In other words, with the bottom left cell only.