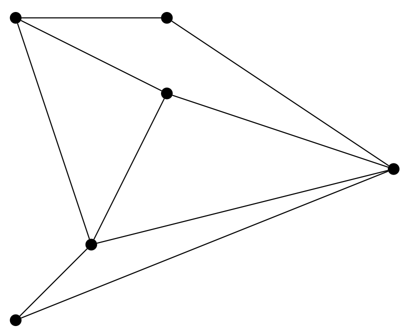
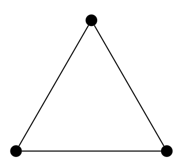
Intermediate Math Circles
Wednesday November 9 2016
Mathematical Games III

1. Cops and Robbers

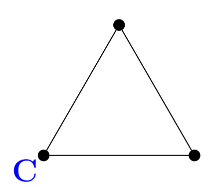
Cops and robbers is a game under the heading “pursuit and evasion”, if you care to look it up on the internet. For our purposes, imagine that there are a bunch of points drawn on a sheet of paper with some of them connected. This is known as a graph. For example,



The game is played between a cop and robber who start on different points. The cop chooses any point to start, then the robber chooses any point to start. Starting with the cop, they take turns moving. A legal move is one which takes them to another point connected directly to the point they are on. They may also choose not to move. A graph is called “cop win” if a cop has a strategy to eventually occupy the same point as the robber. It is called “robber win” if the robber can evade the cop indefinitely. For example, this graph is a cop win:

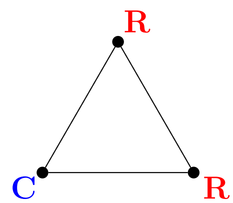


This is because, no matter where the cop starts, for example,

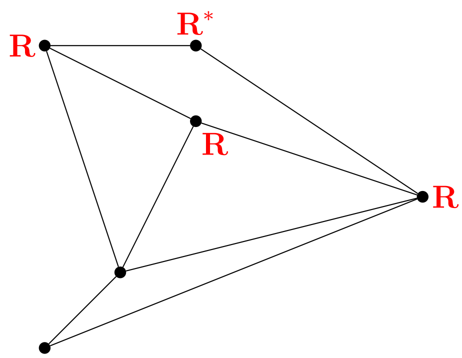


the robber has nowhere safe to go.

The cop can reach the robber at either of the two available starting positions:



The graph we started with is a robber win. It tends to be tricky to explain how the robber can "evade indefinitely". Here is an attempt at an explanation. The robber should start at the one of the four indicated points



which is farthest from where the cop chose to start. That is, if the cop starts on one of those four points, the robber should start at the one "opposite" the cop. Otherwise, the robber should start at the one marked by **R***. The robber's strategy is to always stay opposite the cop when the cop is occupying one of those four points, and if the cop steps away to the two points not marked by an **R** or **R***, the robber should retreat to the point marked by **R***. The cop can never catch the robber if the robber plays this way. Take some time to convince yourself of this.

2. Probability

Probability is a vast topic in mathematics that I hope you have seen already. This will be a short introduction.

Example. Imagine flipping a coin. It should be equally likely that you get heads or tails. We say that the probability of flipping a head is 0.5, and the probability of flipping a tail is 0.5. [This is because 0.5 and 0.5 are the only equal numbers that add up to 1.]

Probability gives us a way to predict the future. Saying that the probability of getting heads is 0.5 really means that if you flip a coin a whole bunch of times (say 1 million or some other large number) you should get heads "about" half of the time.

Example. Two coins are flipped. What is the probability of getting two heads?

Solution. Here is a guess: There are three possible outcomes:

Two heads
Two tails
One of each.

You might guess that the probability of getting heads is one-in-three, or $\frac{1}{3}$ or $0.333\ldots$ since it one of three possible outcomes.
After flipping a pair of coins 300 times, I came up with this:

Two Heads	Two tails	One of each
68	73	159

This doesn't really agree with our prediction. "One of each" occurred about half of the time, and two heads occurred far less than $\frac{1}{3}$ of the time. Keep reading for an explanation.
The possible outcomes after flipping two coins, one after the other, are

$$HH,\ HT,\ TH,\ \text{and},\ TT,$$

so there are actually four possible outcomes, not three. "One of each" shows up in two different ways. The moral of the story is that order matters. To answer the question, we see that of the four possible outcomes, only one of them is HH, so the probability of getting two heads is one-in-four, or 0.25.

The first coin flip and the second coin flip do not have any influence on each-other. We call such events "independent". When events are independent, the probability of both of them happening is the product of the two individual probabilities. Going back to the previous example, the probability of flipping a head on the first flip is 0.5, and the probability of flipping a head the second time is also 0.5. Multiplying the two, we see that the probability of flipping two heads is $0.5 \times 0.5 = 0.25$.

Example. Three coins are flipped. What is the probability of flipping three heads?

Solution. Following the same reasoning as the discussion before the example, each coin flip is independent. The only way to get three heads is to get heads every time. The coin flips are independent, so we just take the product of the individual probabilities to get $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$. Indeed, the possible outcomes are

$$HHH, HHT, HTH, HTT, THH, THT, TTH, TTT.$$

There are 8 in total, and exactly one of them is HHH.

Here is a slightly different example.

Example. Three coins are flipped. What is the probability that either three heads are flipped or three tails are flipped?

Solution. As we saw in a previous exercise, there are 8 possible outcomes. Two of them fit the description "either all heads or all tails", so the probability is $\frac{2}{8}$ or $\frac{1}{4}$.

Example. Two dice are rolled. What is the probability that the sum is 1? How about a 2? What about 3?

Solution. There is no way to roll a 1, since the sum is at least 2. We say that the probability is 0 in this case. To roll a 2, you must roll a 1 both times. There is a $\frac{1}{6}$ chance of rolling a 1 each time, and each roll is independent of the other. Therefore, the probability of rolling a 1 both times is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$. If you think about it, there are 36 equally likely possible outcomes after rolling two dice, and both 1 is just one of them.
Rolling a 3 is a little more interesting. There are two ways of doing it. You can either roll a 1 then a 2, or a 2 then a 1. There are 36 possible outcomes, and two of them have a sum of 3. Therefore, the probability of rolling a sum of 3 is $\frac{2}{36} = \frac{1}{18}$.

Notice that the probability of rolling a 1 followed by a 2 is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$, and the probability of rolling a 2 followed by a 1 is the same, and

$$\frac{1}{36} + \frac{1}{36} = \frac{1}{18},$$

which is the probability from the exercise.
The question "What is the probability of rolling a sum of 3?" from the previous exercise could be rephrased as "What is the probability of rolling either 1 then 2 **or** 2 then 1?". In this situation, when there is an "or" in the question, it is safe to just take the sum of the two probabilities provided the two outcomes can't occur at the same time.