Intermediate Math Circles
Wednesday November 9 2016
Problem Set 6

1. Find a winning strategy for the following game: Two players alternate removing stones from a pile, and the player who removes the last stone wins (as usual). The only restriction is that players may only remove a power of 2 on a move (that is, 1, 2, 4, 8, etc.). Find all starting sizes for which the second player has a winning strategy. How does the strategy change if you are only allowed to remove powers of 3? What about powers of 5?

   Solution

   The winning strategy is to always leave your opponent with a multiple of 3. If you leave your opponent with a multiple of 3, they can not leave you with a multiple of 3. To see why, imagine that your opponent has to pick from a pile of $3k$ stones and they remove 2^n stones. You are left with $3k - 2^n$. This can not be a multiple of 3, because $3k$ is a multiple of 3 and 2^n is not. This means that if you leave your opponent with a multiple of 3, they can not leave you with a multiple of 3. If you are faced with a number which is not a multiple of 3, it is either 1 or 2 more than a multiple of 3. You can remove 1 or 2, depending on the situation, to leave your opponent with a multiple of 3.

2. The game in this problem is the same as the previous one except players are only allowed to remove perfect squares, not powers of 2. That is, the legal numbers of stones to remove are $1, 4, 9, 16$, etc. Show that there is no largest pile size for which the second player has a winning strategy. [This one is quite tricky.]

   Solution

   See the October 26 notes for notation on "Previous player win" and "Next player win". Numbers on which Player 2 has a winning strategy are those which are previous player win, and they are next player win, otherwise. Keep in mind that a number is previous player win if every legal move leaves the number of stones in the pile as a next player win. We will argue that there is no largest previous player win by showing that for any given previous player win, there is a larger one. Suppose k is previous player win. Consider the number $k^2 + k + 1$. If $k^2 + k + 1$ is a previous player win, we have found a bigger previous player win, and the argument is over. Otherwise, assume that $k^2 + k + 1$ is a next player win. For a number to be a next player win, it means there is a move to a previous player win. If $(k + 1)^2$ stones are removed, then the number of remaining stones is $k^2 + k + 1 - (k + 1)^2 = k^2 + k + 1 - k^2 - 2k - 1 = -k$, which is a negative number, so removing $(k + 1)^2$ stones is not a legal move. Of course, removing more than $(k + 1)^2$ stones is also illegal. Therefore, the only legal move is to remove at most k^2 stones. Since $k^2 + k + 1$ is a next player win, there is some $n \leq k$ so that $k^2 + k + 1 - n^2$ is a previous player win. Since $k^2 \geq n^2$, $k^2 + k + 1 - n^2 = (k^2 - n^2) + k + 1 \geq k + 1 > k$, so $k^2 + k + 1 - n^2$ is a previous player win that is bigger than k. We have shown that either $k^2 + k + 1$ is a previous player win, or $k^2 + k + 1 - n^2$ is a previous player win. Both are bigger than k, so either way, we have found a bigger previous player win. We conclude that there is no largest previous player win.

3. Find a winning strategy for a single pile subtraction game (like (1) and (2)) where the players are allowed to remove 1, 2, or 6 stones. Choose your own set of allowed numbers and find a winning strategy.

Solution

We know that the piles with 1, 2, and 6 stones are next player win. We can start to fill out a table as follows:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
N	N				N																		

This means that the pile with 3 stones is a previous player win, since legal moves can only leave 1 stone or 2 stones, which are next player wins. Of course, this gives 4, 5, and 9 legal moves to 3, which is a previous player win, so they are all next player wins. The table continues as follows:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
N	N	P	N	N	N			N															

Because of how the table has been filled out, 7 can not have a move to a previous player win, so it must be next player win. This gives legal moves from 8, 9, and 13 to a previous player win, so they are next player wins.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
N	N	P	N	N	N	P	N	N				N											

Continuing, we can always fill out the first empty space with a P, then fill out the spaces 1, 2, and 6 to its right by N. The chart fills out as follows:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
N	N	P	N	N	N	P	N	N	P	N	N	N	P	N	N	P	N	N	N	P	N	N	P

The pattern can now be seen to emerge. Notice that the multiples of 7 are all previous player wins, and the numbers 3, 10, 17, and 24 are previous player wins. The second sequence is exactly the set of numbers that are three more than a multiple of 7. The winning strategy is as follows: If you are faced with a pile of stones that has either a multiple of 7 or 3 more than a multiple of 7, you are out of luck. Otherwise, remove enough stones to leave your opponent with a multiple of 7, or 3 more than a multiple of 7.

4. Call a number **evil** if there are exactly two 1s in its binary expansion. For example, 3, 5, and 20 are evil since their binary expansions are 11, 101, and 10100, which each have exactly two 1s. Find the sum of all evil numbers which are less than 1024. [Hint: First, try to find the sum of all evil numbers less than 8, less than 16, and less than 32 to try to find a pattern.]

   Solution

   Following the hint, the binary expansion of 8 is 1000, so the evil numbers less than 8 are $110, 101$, and 11, which are 6, 5, and 3. Their sum is 14. Looking a bit more closely, the numbers are $2^2+2^1, 2^2+2^0$, and 2^2+2^0. Their sum is $2^2+2^1+2^2+2^0+2^1+2^0 = 2(2^2+2^1+2^0) = 2(7) = 14$. The evil numbers less than 16 are, in binary, $1100, 1010, 1001, 110, 101, 11$. The numbers in decimal are $12, 10, 9, 6, 5, 3$, the sum of which is 45. Again, looking a little closer, the numbers are $2^3+2^2, 2^3+2^1, 2^3+2^0, 2^2+2^1, 2^2+2^0$, and 2^1+2^0, the sum of which is

$$2^3+2^2+2^3+2^1+2^3+2^0+2^2+2^1+2^2+2^0+2^1+2^0 = 3(2^3+2^2+2^1+2^0) = 3(8+4+2+1) = 2(15) = 45.$$

   There is a pattern here. For a number to have two 1s in its binary expansion, it needs to be the sum of two different powers of 2. For evil numbers less than 32, we have

$16+8$
$16+4$
$16+2$
$16+1$
$8+4$
$8+2$
$8+1$
$4+2$
$4+1$
$2+1$

.

The sum contains 4 16s, 4 8s, 4 4s, 4 2s, and 4 1s, so it is $4(16+8+4+2+1) = 4(2^4+2^3+2^2+2^1+2^0) = 4(31) = 124$. In general, the evil numbers less than 2^n are

$2^{n-1}+2^{n-2}$
$2^{n-1}+2^{n-3}$
$\vdots$
$2^{n-1}+2^0$
$2^{n-2}+2^{n-3}$
$2^{n-2}+2^{n-4}$
$\vdots$
$2^{n-2}+2^0$
$\vdots$
2^3+2^2
2^3+2^1
2^3+2^0
2^2+2^1
2^2+2^0
2^1+2^0

If you count carefully, each power of 2 occurs $n-1$ times, so the sum is

$$(n-1)(2^{n-1}+2^{n-2}+\cdots+2^1+2^0).$$

Now for a neat trick. Suppose $S = 2^{n-1}+2^{n-2}+\cdots+2+1$. Then

$$\begin{aligned} 2S &= 2(2^{n-1}+2^{n-2}+\cdots 2^1+2^0) \\ &= 2^n+2^{n-1}+\cdots+2^2+2, \end{aligned}$$

so

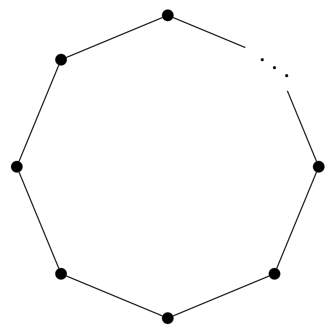
$$S = 2S - S = (2^n+2^{n-1}+\cdots+2^2+2) - (2^{n-1}+2^{n-2}+\cdots+2+1) = 2^n - 1.$$

Using this, we get that the sum of the evil numbers less than 2^n is $(n-1)(2^n-1)$. When $n=3$, this is $(3-1)(2^3-1) = (2)(7) = 14$, and when $n=4$, it is $(4-1)(16-1) = (3)(15) = 45$, so it agrees with the earlier sums. Since $1024 = 2^{10}$, the sum of the evil numbers less than 1024 is $(10-1)(2^{10}-1) = (9)(1023) = 9207$.

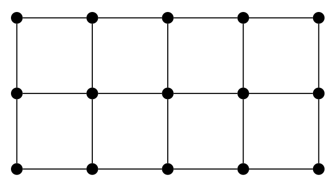
5. Determine whether or not the following are cop win or robber win.

   (a) A path. That is,

   (b) A cycle. That is,

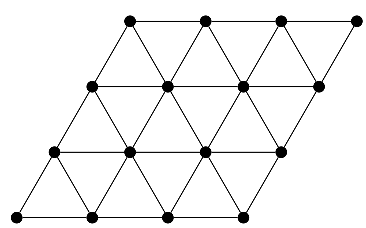
   

   (c) A rectangular grid. For example, a 3×5 rectangular grid looks like

   You might think this is a 2×4 grid, which is fine.

   (d) A triangular grid. For example,

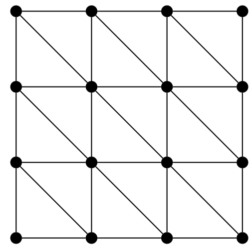
Solution

(a) A path is cop win. The strategy is simply to always move towards to robber. The robber can't jump over the cop, so they must eventually occupy the same point.

(b) As long as the cycle has at least 4 points, the robber can evade capture indefinitely. The robber should choose a position that is at least two steps away from the cop in either direction. This can be done since the cycle has at least 4 points. After that, the robber should always moves in the same direction as the cop.

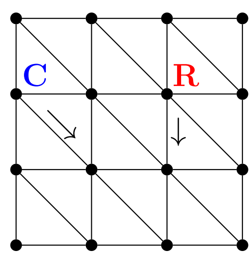
(c) This one is actually a lot easier than it looks. The robber should choose a corner that is not adjacent (connected by a line) to where the cop started. For simplicity, assume this is the top right corner. The robber can stay in the top right 2×2 corner and evade capture forever. This is done by staying on the top right point when the cop is not in this 2×2 block, and moving to the point in the 2×2 block that is two steps away from the cop, should the cop enter the 2×2 block.

(d) It is helpful to skew the grid and redraw it like this:



This one is cop win, and here is the strategy. Since the robber can not move in the same vertical direction indefinitely, the cop can eventually achieve a position in the same row as the robber. For example, if the robber starts off higher than the cop (not necessarily in the same column), then the cop should move up until they are in the same row. Since the robber can't move up forever, they must eventually move down or horizontally. At this time, the cop can reduce the vertical distance between them.

Let's now suppose that the cop has achieved this. That is, the cop and robber are in the same row. If they are in the same column as well, the cop has already won. Otherwise, the robber is either to the right or to the left of the cop. Let's deal with the case where the robber is to the right of the cop. The other case is similar. If it is the cop's turn, they should move to the right and reduce the distance between them. The robber now has 6 options. If they choose to move right, diagonally down-right, or up, the cop should do the same. This will return them to a position where they are the same distance apart as when it was the robber's turn to move, and they will also be in the same row again. If the robber moves down, the cop should move diagonally down-right. This reduces the distance between them. Below is an example:



If the robber moves left, the cop should move right, which (more obviously) decreases the distance between the two. Finally, if the robber moves diagonally up-left, the cop should move up. This will also decrease the distance between them. If the robber chooses not to move, the cop should move right and reduce the distance between them.

To finish off the argument, you should convince yourself that the robber can not move right, up, and diagonally down-right forever. Eventually, they will have to move down, left, diagonally up-left, or stay put. In any of these cases, there is a move for the cop to reduce the distance between the two. furthermore, the cop can move so that the distance never increases (after a cycle of one move each). Eventually, if the distance keeps going down, it will be 0, which means the cop has won!

6. A 6 sided die is rolled twice. What is the probability that the product of the two rolls is

   (a) equal to 5,

   (b) equal to 6,

   (c) equal to 7,

   (d) more than 6,

   (e) more than 20.

Solution

Let's let (a, b) be the event where a is rolled first and b is rolled second. For example, $(2, 3)$ means 2 was rolled, then a 3 was rolled. $(1, 1)$ means that 1 was rolled both times.

(a) Since 5 is a prime number, the only ways to achieve this are $(1, 5)$ and $(5, 1)$. Therefore, the probability is $\frac{2}{36}$ since there are 2 rolls with a product of 5, and 36 possible rolls (of two dice).

(b) For this one, it can happen in 4 possible ways: $(1, 6), (6, 1), (2, 3)$, and $(3, 2)$. That makes 4 ways to roll a product of 6. Again, there are 36 possible rolls, so the probability is $\frac{4}{36} = \frac{1}{9}$.

(c) Since 7 is a prime number, the only way that it could happen is if a 1 and a 7 were rolled. There is no way to roll a 7, so there are 0 rolls that have a product of 7. The probability is therefore $\frac{0}{36} = 0$.

(d) Again, let's count the number of rolls that have a product *less than or equal to* 6. The rolls with a product of $1, 2, 3, 4, 5$, or 6 are:

$$(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (4, 1), (5, 1), (6, 1).$$

There are 14 in the list, which means the other $36 - 14 = 22$ must have a product greater than 6. Therefore, the probability of rolling a product greater than 6 is $\frac{22}{36} = \frac{11}{18}$.

(e) This time, we will count the number of pairs that have a product greater than 20. If you roll a and b and $a \times b > 20$, then you must have that at least one of a and b is bigger than 4. This is because if they are both smaller than or equal to 4, their product will be no more than $4 \times 4 = 16$. By similar reasoning, if one of a and b is 3, then $a \times b$ can not be bigger than $3 \times 6 = 18$. Therefore, both dice are at least 4. The possible products are

$$4 \times 4, 4 \times 5, 4 \times 6, 5 \times 5, 5 \times 6.$$

The products 4×4 and 4×5 are 16 and 20, respectively, which are not bigger than 20. The rest are bigger than 20, so the pairs whose product is greater than 20 are

$$(4, 6), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6).$$

There are 6 pairs, so the probability of rolling a product bigger than 20 is $\frac{6}{36} = \frac{1}{6}$.

7. A 6-sided die is rolled three times. What is the probability that the sum of the three rolls is more than

   (a) 3,
   (b) 4,
   (c) 15,
   (d) 17,
   (e) 9.

   Solution

   Similar to the previous problem, let (a, b, c) denote the event where a was rolled, then b, then c.

   (a) For this one, every roll except (1,1,1) has a sum greater than 3. There are $6 \times 6 \times 6 = 216$ possible rolls, so the probability is $\frac{215}{216}$.

   (b) This question is asking what the probability of rolling something that is not 3 or 4, since you can't roll a total of 1 or 2. We can figure this out by counting how many ways there are to roll a 3 or 4 and subtracting from the total of 216. There is only one way to get a sum of 3, which is $(1, 1, 1)$. To roll a 4, you must roll one 2 and two 1s. This can happen in three different ways: $(2, 1, 1), (1, 2, 1)$, and $(1, 1, 2)$. Therefore, there are 4 ways to roll *at most 4*, so there must be $216 - 4 = 212$ ways to roll greater than 4. The probability, therefore, is $\frac{212}{216} = \frac{1}{54}$.

   (c) This one might seem tricky, but it still just comes down to carefully counting how many different rolls will give a sum of $16, 17$, or 18 (18 is the largest possible sum). Suppose, at some point, a 4 is rolled. If the total is going to be 16, 17, or 18, the other two rolls have to be 6 since this is the only way for the total to reach as much as 16. This gives $(4, 6, 6), (6, 4, 6)$, and $(6, 6, 4)$. We have three possibilities so far. There is no way to do it if a 3 is rolled at any point since the total will be no more than $3 + 6 + 6 = 15$. Therefore, all other possibilities consist of only 5s and 6s. There are 8 such rolls (2 choices in each of the three positions, so the number of rolls consisting of only 5s and 6s is $2 \times 2 \times 2 = 8$). However, $(5, 5, 5)$ has a sum of 15, so it should not be counted. The rest have a total of at least 16, so there are 7 more. With the three from earlier, there are a total of 10 ways to roll a sum greater than 15, so the probability is $\frac{10}{216} = \frac{5}{108}$.

   (d) This one is easy. The only way to roll more than 17 is to roll 18, which only happens with the sequence of rolls $(6, 6, 6)$. Therefore, the probability of rolling more than 17 is $\frac{1}{216}$.

(e) This one is tough. We want to count the number of triples, (a, b, c), where a, b, and c are between 1 and 6 (including 1 and 6) and $a + b + c \leq 9$. We can then subtract from the total. Here is one way to organize the counting. We will break the counting up into 6 cases: when the first roll is 1, when the first roll is 2, and so on. When the first roll is 1, we need the sum of the next two rolls to be no more than 8. So we need to answer the question "how many ways are there to get a sum of no more than 8 when rolling 2 dice?" Well, the ways to get more than 8 are $(3, 6), (6, 3), (4, 5), (5, 4), (4, 6), (6, 4), (5, 5), (5, 6), (6, 5)$ and $(6, 6)$. There are 10, so there are $36 - 10 = 26$ ways to roll no more than 8. In summary, we get 26 triples (a, b, c) where $a = 1$ and $a + b + c \leq 9$. Now we play the same game for 2. That is, count the number of ways to roll no more than a sum of 7 with 2 dice. There are 21 ways, so there are 21 triples starting with 2. Starting with 3, there are 15, there are 10 starting with 4, 6 starting with 5, and 3 starting with 6. The total number of triples is $26 + 21 + 15 + 10 + 6 + 3 = 81$. Therefore, the number of ways to roll more than 9 is $216 - 81 = 135$, and the probability of rolling more than 9 is $\frac{135}{216} = \frac{5}{8}$.

8. The digits 1,2,3,4, and 5 are arranged to form a 5-digit number. What is the probability that the number is

(a) even,

(b) a multiple of 5,

(c) a multiple of 11.

(d) Answer (a), (b), and (c) with the digits 1 through 6, rather than 1 through 5.

Solution

The number of ways to order k different numbers is $1 \times 2 \times 3 \times \cdots \times k$. To see why this is true, imagine choosing an order from left to right. There are k choices for the first number. When choosing the next number, there are $k-1$ choices, because one of the numbers is gone. For similar reasoning, there are $k-2$ choices for the third number, and so on. The number of total choices is their product, so it is $k \times (k-1) \times (k-2) \times \cdots \times 2 \times 1 = 1 \times 2 \times \cdots \times k$.

(a) The number will be even if the last digit is either 2 or 4. If the last digit is chosen to be 2, the digits 1,3,4, and 5 can be arranged in any order to the left of the 2. There are $4 \times 3 \times 2 \times 1 = 24$ ways to do this. If the last digit is 4, there are 24 ways to arrange the digits 1,2,3, and 5. This gives a total of 48 ways to arrange the digits to form an even number. There are a total of $1 \times 2 \times 3 \times 4 \times 5 = 120$ ways to arrange the digits, so the probability of the number being even is $\frac{48}{120} = \frac{2}{5}$. This makes sense, since it should be equally likely that any of the 5 digits appear in the units place, and 2 out of 5 of them are even.

(b) The only way for the number to be a multiple of 5 is for the last digit to be 5. Following the reasoning at the end of the last problem, we might guess that the probability is 1 in 5 since one of the five possible units digits is 5. Counting the numbers that are multiples of 5 comes down to counting how many ways there are to order the digits $1, 2, 3$, and 4. As we have seen before, there are 24 such ways, so the probability is $\frac{24}{120} = \frac{1}{5}$.

(c) This one is much more tricky. Remember from last time that the number, $abcde$, is a multiple of 11 exactly when $a - b + c - d + e$ is a multiple of 11. Rearranging, this means $(a + c + e) - (b + d)$ is a multiple of 11. Keep in mind that we need to use each of the digits 1 through 5 exactly once, so there aren't many multiples of 11 that it can possibly be. The biggest $(a + c + e) - (b - d)$ can possibly be is when the things that are added are the largest numbers, and the things that are subtracted are the smallest numbers. That is, the largest it can be is $3 + 4 + 5 - 1 - 2 = 9$. Similarly, the smallest it can possibly be is $1+2+3-4-5 = -3$. The only multiple of 11 between -3 and 9 is 0, so $(a + c + e) - (b + d)$ must be 0 for $abcde$ to be a multiple of 11. Now for a trick. Note that $1 + 2 + 3 + 4 + 5 = 15$, which is an odd number, and suppose $(a + c + e) - (b + d) = 0$. Then $a + c + e = b + d$, and now we add $b + d$ to both sides to get $a + c + e + b + d = 2(b + d)$. The sum on the left is $a + b + c + d + e$, which we know is $1 + 2 + 3 + 4 + 5$ in some order. Therefore, $15 = 2(b + d)$, but this says that 15 is an even number! This is nonsense, so there is no way that $(a + c + e) - (b + d) = 0$. We conclude that there is no way to arrange the digits 1,2,3,4 and 5 to make a multiple of 11, which is kind of cool and possibly surprising. To answer the question, the probability is 0.

(d) In order to be even, it has to end with 2, 4, or 6. You can do a similar computation to the previous one, or note that these are half of the possible ending values. You might guess that the probability of being even is $\frac{1}{2}$, and you are right! Try a similar calculation to (a) to confirm this. For this one, again, the only way for it to be a multiple of 5 is for it to end in 5. Since there are 6 possible ending digits and they should all be equally likely, we might guess that the probability is $\frac{1}{6}$. Indeed, there are 120 ways to order 1,2,3,4, and 6, and there are $1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$ possible ways to order all digits, so the probability is $\frac{120}{720} = \frac{1}{6}$.

The question of multiples of 11 is similar to the previous one. Again, the number $abcdef$ is a multiple of 11 exactly when $a-b+c-d+e-f = (a+c+e)-(b+d+f)$ is a multiple of 11. Since the numbers a, b, c, d, e, and f are the number from 1 through 6 in some order, the largest this can be is $(4 + 5 + 6) - (1 + 2 + 3) = 9$, and the smallest it can be is -9. Therefore, if it is going to be a multiple of 11, it must be that $(a + c + e) - (b + d + f) = 0$, or $a + c + e = b + d + f$. Like part (c), this means that $a + b + c + d + e + f = 2(b + d + f)$, but $1 + 2 + 3 + 4 + 5 + 6 = 21$ is odd, so the answer is again, 0.

9. Player A and Player B are playing a game where they are flipping a coin repeatedly. The game stops when one of these four things happens:

   (i) A head is flipped first, in which case Player A wins.

   (ii) Two heads are flipped in a row, in which case Player A wins.

   (iii) Two tails are flipped in a row, in which case Player B wins.

   (iv) After the coin has been flipped 5 times, if none of (i)-(iii) has occurred, the game ends and it is a draw.

   This game is entirely based on chance, so there is no winning strategy possible. Here are some questions about the game.

   (a) What is the probability that the game ends in a draw?

   (b) What is the probability that Player A wins?

   (c) What is the probability that Player B wins?

   (d) Answer (a) and (b) when the game is changed so that the "5" in (iv) is changed to "100". You answer can be a (long) sum of numbers. [Hint: This one is also hard.]

Solution

(a) The only way for the game to end in a draw is if tails is flipped first, then heads, then tails, then heads, then tails. That is, the sequence is $THTHT$. This happens with probability $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$.

(b) The sequences that result in a Player A win are H, THH, and $THTHH$. There is a $\frac{1}{2}$ chance that the first will occur, there is a $\frac{1}{2^3}$ chance that the second will occur, and a $\frac{1}{2^5} = \frac{1}{32}$ chance that the third will occur. None of these can occur at the same time, so the probability that Player A wins is $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} = \frac{21}{32}$.

(c) The winning sequences for Player B are TT and $THTT$. These occur with probability $\frac{1}{4}$ and $\frac{1}{16}$, so the probability is $\frac{1}{4} + \frac{1}{16} = \frac{5}{16}$. Notice that $\frac{1}{32} + \frac{21}{32} + \frac{5}{16} = 1$. This is no coincidence. The three things that can happen are Player A win, Player B win, and a draw. The sum of the probabilities of all possible outcomes is always 1.

(d) Similar to (a), the only way for the game to end in a draw is for the sequence $\underbrace{THTHTH\cdots THTH}_{100}$ to be flipped. This occurs with probability $\frac{1}{2^{100}}$, which is 1 divided by a roughly 30 digit number, so very close to 0. This means it is almost impossible for the game to end in a draw.

Similar to (b), the winning sequences for Player A are $H, THH, THTHH, THTHTHH$, and so on. The last one will have 99 coin flips, so the probability of the individual sequences is $\frac{1}{2}$, $\frac{1}{8}$, $\frac{1}{2^5}$, $\frac{1}{2^7}$, $\ldots$, $\frac{1}{2^{99}}$. Therefore, the probability of a Player A win is

$$\frac{1}{2^1} + \frac{1}{2^3} + \cdots + \frac{1}{2^{99}}.$$

You can stop here if you like, but it is worth manipulating this a little bit. You can carefully write this as one fraction to get

$$\frac{2^{98}+2^{96}+2^{94}+\cdots+2^2+1}{2^{99}}=\frac{1+4+4^2+4^3+\cdots+4^{49}}{2^{99}}.$$

Using a similar trick to problem 4, if we set $S=1+4+4^2+\cdots+4^{49}$, we get $4S=4+4^2+4^3+\cdots+4^{50}$, so

$$4S-S=(4+4^2+\cdots+4^{50})-(1+4+4^2+\cdots+4^{49})=4^{50}-1.$$

Therefore, $S=\dfrac{4^{50}-1}{3}$. Rewriting 4^{50} as 2^{100}, the probability of a Player A win is

$$\frac{2^{100}-1}{3\cdot 2^{99}}=\frac{2^{100}}{3\cdot 2^{99}}-\frac{1}{3\cdot 2^{99}}=\frac{2}{3}+\frac{1}{3\cdot 2^{99}}.$$

The second part of the final sum is very, very small, so the probability that Player A wins is extremely close to $\frac{2}{3}$.

Using the observation that the sum of the probabilities that Player A wins, Player B wins, and it is a draw sum to 1, we get that the probability that Player B wins is

$$\begin{aligned}
& 1-\frac{2^{100}-1}{3\cdot 2^{99}}-\frac{1}{2^{100}}\\
=\ & \frac{3\cdot 2^{100}}{3\cdot 2^{100}}-\frac{2^{101}-2}{3\cdot 2^{100}}-\frac{3}{3\cdot 2^{100}}\\
=\ & \frac{3\cdot 2^{100}-2^{101}+2-3}{3\cdot 2^{100}}\\
=\ & \frac{(1+2)2^{100}-2^{101}-1}{3\cdot 2^{100}}\\
=\ & \frac{2^{100}+2^{101}-2^{101}-1}{3\cdot 2^{100}}\\
=\ & \frac{2^{100}-1}{3\cdot 2^{100}}
\end{aligned}$$

By the way, this is very close to $\frac{1}{3}$. Here is another question: If there are no draws, that is, they play until somebody wins, what is the probability that Player A wins? This is a strange question because, in theory, the game could go on forever. Realistically, this is not going to happen. Playing the game with a maximum of 100 flips is very close to playing it forever, so the probability should be $\frac{2}{3}$. Can you think of a more intuitive reason why Player A is twice as likely to win as Player B?