Intermediate Math Circles
Wednesday 12 October 2016
Geometry II: Side Lengths

Problems From Last Week

We took up four or five problems from last week. Complete solutions can be found on our website at http://www.cemc.uwaterloo.ca/events/mathcircle_presentations.html.

Congruent Triangles

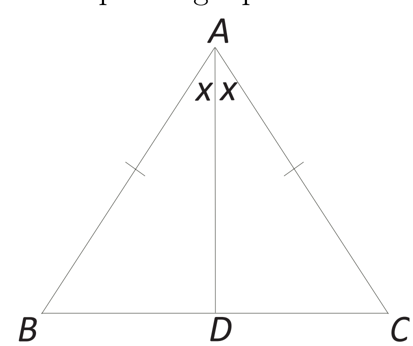
Two triangles are called *congruent* if corresponding side lengths and corresponding angles are all equal. In other words, the triangles are equal in all respects. Sometimes, fewer than these 6 equalities are necessary to establish congruence. Some ways to determine that two triangles are congruent:

- Side-Side-Side (SSS)
- Side-Angle-Side (SAS)
- Angle-Side-Angle (ASA)
- Right Angle-Hypotenuse-Side (RHS)

Once two triangles are proved to be congruent, all of the other corresponding equalities follow.

Prove:

If a triangle is isosceles, then the angles opposite the equal sides are equal.



Proof:

Construct isosceles $\triangle ABC$ so that $AB = AC$. We are required to prove $\angle ABC = \angle ACB$.

Construct AD, the angle bisector of $\angle BAC$.
It follows that $\angle BAD = \angle DAC$.

In $\triangle BAD$ and $\triangle CAD$, $AB = AC$, $\angle BAD = \angle CAD$ and AD is common. Therefore, $\triangle BAD \cong \triangle CAD$. It follows that $\angle ABC = \angle ACB$, as required.

From the triangle congruency, we get two additional results.

First, $BD = DC$. This means that the angle bisector between the two equal sides of an isosceles triangle bisects the base.

Second, $\angle BDA = \angle CDA$. But $\angle BDA$ and $\angle CDA$ form a straight angle.
So $180° = \angle BDA + \angle CDA = 2\angle BDA$. It then follows that $\angle BDA = 90°$.

Therefore, $\angle BDA = \angle CDA = 90°$ and $BD = DC$.

This means that the angle bisector constructed between the two equal sides of an isosceles triangle right bisects the base. In fact, this bisector is also an altitude. This result is **not** true for the other angle bisectors in an isosceles triangle.

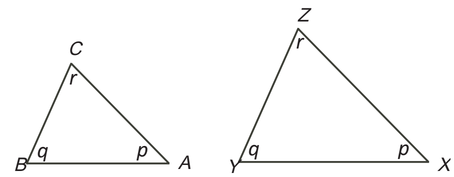
Similar Triangles

• Two triangles are called *similar* if corresponding angles are equal. If two triangles are similar, then the corresponding pairs of sides are in a constant ratio.

In this example, $\angle A = \angle X$ and $\angle B = \angle Y$ and $\angle C = \angle Z$. Therefore, $\triangle ABC \sim \triangle XYZ$.

Then $\dfrac{AB}{XY} = \dfrac{AC}{XZ} = \dfrac{BC}{YZ}$.

In other words, the triangles are "scaled models" of each other.

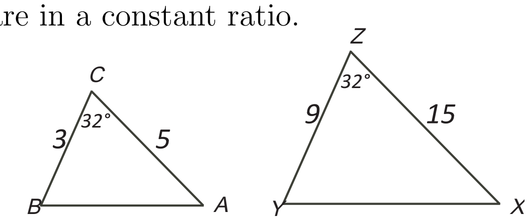


• Two triangles are also similar if two pairs of corresponding sides are in constant ratio and the angles between the sides are equal.

Once similarity is shown then the corresponding pairs of sides are in a constant ratio.

In this example, $\dfrac{CB}{ZY} = \dfrac{CA}{ZX} = \dfrac{1}{3}$ and $\angle C = \angle Z$..

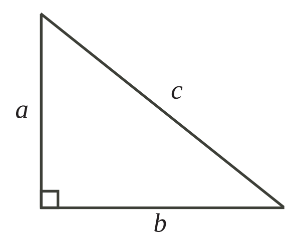
As a result of similarity, $\angle B = \angle Y$, $\angle A = \angle X$ and $\dfrac{BA}{YX} = \dfrac{1}{3}$.



The Pythagorean Theorem:

In a right-angled triangle, the *hypotenuse* is the longest side and is located opposite the 90° angle. In any right-angled triangle, the square of the hypotenuse equals the sum of the squares of the other two sides.

In the triangle illustrated to the right, $a^2 + b^2 = c^2$.



Proofs of The Pythagorean Theorem:

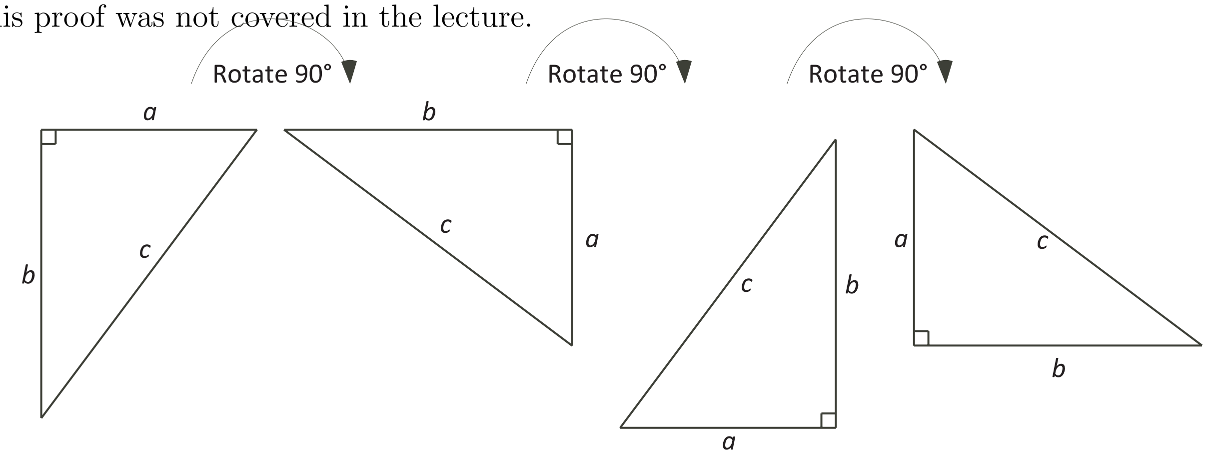
If you do an internet search you will discover many different proofs of the Pythagorean Theorem. If you go to the link http://www.cut-the-knot.org/pythagoras/index.shtml#84, you will find 98 of the proofs grouped together. We will present three proofs here.

Proof #1:

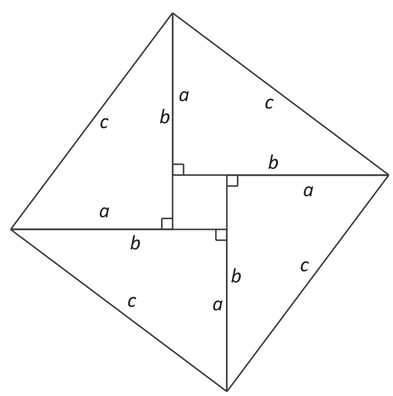
The first proof presented was a visual proof. It will not be included in these notes. It can be viewed on the video.

Proof #2:

This proof was not covered in the lecture.



Starting with the leftmost right triangle, rotate 90° to the right to create the second triangle. Rotate the second triangle 90° to the right to create the third triangle and rotate the third triangle 90° to the right to create the fourth triangle. This process creates four congruent right triangles. We will now reposition the four right triangles to create the following figure.



The figure is a square with sides of length c. We can see that the sides are each length c but are the corners 90°? Let the angle between side b and side c be α. Then the angle between side a and side c is $90 - \alpha$. Each corner then consists of an α and $90 - \alpha$. The angle at each corner is $\alpha + 90 - \alpha = 90°$.

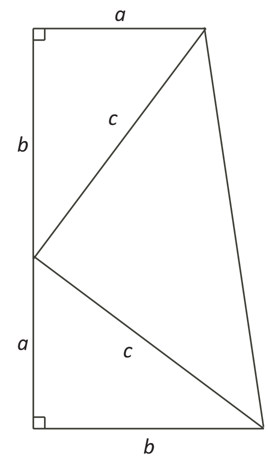
The figure in the centre is a square. Each side is $b - a$ units.

The large square is made up of four congruent triangles and a smaller square. We will construct an equation using area.

$$\begin{aligned}
\text{Area of Large Square} &= \text{Area of 4 triangles} + \text{Area of inner square} \\
c^2 &= 4 \times \left(\frac{a \times b}{2}\right) + (b - a) \times (b - a) \\
c^2 &= 2ab + (b^2 - 2ab + a^2) \\
c^2 &= b^2 + a^2
\end{aligned}$$

This is not the only way to arrange the triangles. A figure can be created with a large outer square of side length $a + b$ and a smaller square of side length c inside along with the four triangles. This will be left as an exercise for the student to pursue.

Proof #3:



This proof is attributed to James Garfield, the twentieth President of the United States. He basically takes the first and fourth triangles from our group of four triangles and stacks them on top of each other as shown.

At the point where the three triangles meet a straight line is formed. Let the angle between side b and side c be α. Then the angle between side a and side c is $90-\alpha$. The remaining angle between the two sides of length c is $180-\alpha-(90-\alpha)=90°$.

The large figure is a trapezoid that contains three right angled triangles. (The justification that the large figure is a trapezoid is straight forward and is not included here.) As in proof #2 we can form an area equation.

$$\begin{aligned}
\text{Area of Trapezoid} &= \text{Area of Two congruent triangles} + \text{Area of Isosceles triangle}\\
\frac{h\times(a+b)}{2} &= 2\times\left(\frac{a\times b}{2}\right)+\frac{c\times c}{2}\\
\frac{(a+b)\times(a+b)}{2} &= 2\times\left(\frac{a\times b}{2}\right)+\frac{c\times c}{2}\\
(a+b)(a+b) &= 2ab+c^2, \qquad \text{after multiplying through by 2}\\
a^2+2ab+b^2 &= 2ab+c^2\\
a^2+b^2 &= c^2
\end{aligned}$$

A *Pythagorean Triple* is a triple (a,b,c) of positive integers with $a^2+b^2=c^2$. The most familiar Pythagorean Triple is $(3,4,5)$.

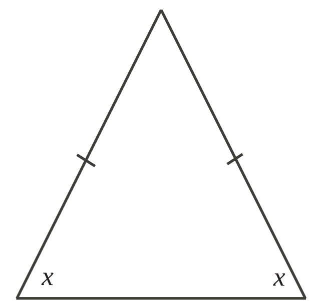
The following chart illustrates several Pythagorean triples. The smallest side length is an odd number.

a	b	c
3	4	5
5	12	13
7	24	25
9	40	41
11	60	61

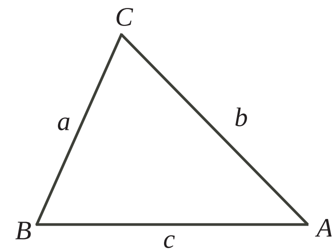
Look for patterns in the table. For example, b and c are consecutive integers, b is even and c is odd. The sum $b+c$ appears to be a perfect square. Can you predict the triple in which the smallest number is 13? Can you predict a formula for generating any Pythagorean Triple with a, the smallest number, an odd number ≥ 3.

We can prove that, for n an odd integer ≥ 3, then $(n, \frac{n^2-1}{2}, \frac{n^2+1}{2})$ is a Pythagorean Triple. This proof will be left for the student.

If a triangle has two angles equal, then the two opposite sides are equal. That is, the triangle is isosceles. The proof of this is left for the student.

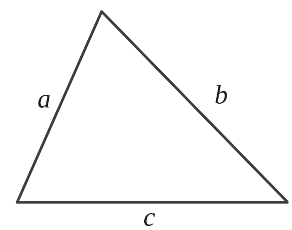


Earlier in these notes we proved the base angle theorem for isosceles triangles that states: if a triangle has two equal sides, then the two opposite angles are equal. The above statement is called a *converse*. When a statement and its converse are both true, we can state them together using *if and only if* (*IFF* for short). The *Isosceles Triangle Theorem* can be stated: A triangle has two equal sides IFF it has two equal angles.



If $\angle A < \angle B$, then $a < b$.

If $a < b$, then $\angle A < \angle B$.



If a, b and c are the side lengths of a triangle, the *Triangle Inequality* tells us that $b+c > a$ and $a+c > b$ and $a+b > c$.

There are two kinds of special triangles.
The first has angles 45°, 45° and 90°.
The second has angles 30°, 60° and 90°.
The side lengths are shown on the diagrams below when the shortest side is 1.
These can be scaled by any factor.