



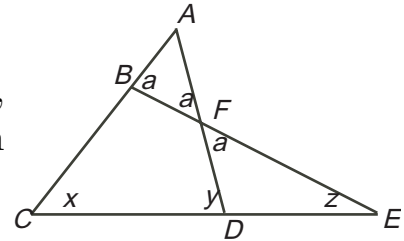
Intermediate Math Circles

Wednesday October 19 2016

Geometry III: Circles

Looking back at Problem Set 1 Question 11

In the figure shown, $AB = AF$ and ABC , AFD , BFE , and CDE are all straight lines. Determine an equation relating x , y and z .



In $\triangle ABF$, since $AB = AF$ then $\angle ABF = \angle AFB = a$.

Since $\angle AFB$ and $\angle DFE$ are opposite angles, then $\angle AFB = \angle DFE = a$.

$\angle ABE$ is exterior to $\triangle BCE$. Therefore, $a = x + z$. (1)

$\angle CDF$ is exterior to $\triangle DFE$. Therefore, $y = a + z$. (2)

Substitute for a from (1) into (2). Therefore, $y = x + z + z$.

Rearranging, $x - y + 2z = 0$.

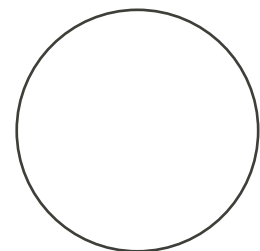
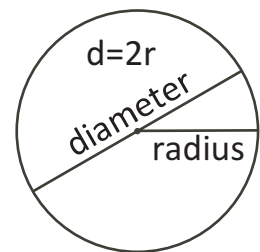
Problems from Problem Set 2?

Full solutions are available at

http://www.cemc.uwaterloo.ca/events/mathcircle_presentations.html.

What do we know about circles?

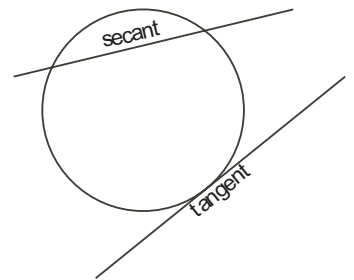
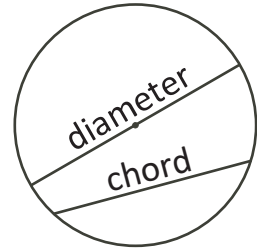
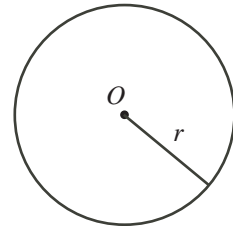
- Circles are round.
- Diameter = $2 \times$ radius.
- $A = \pi r^2$
- $C = \pi d = 2\pi r$.





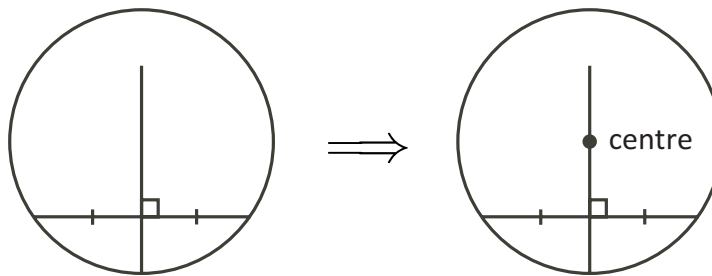
Definitions

- A *circle* is a set of points in 2-space that are all equidistant from a fixed point. The fixed distance is called the *radius* and the fixed point is called the *centre*.
- A *chord* is a line segment with its endpoints on the circumference of a circle.
- A *diameter* is a chord that passes through the centre of a circle.
- A *tangent* is a line (or line segment) that touches a circle in exactly one point.
- A *secant* is a line that intersects a circle in two points.

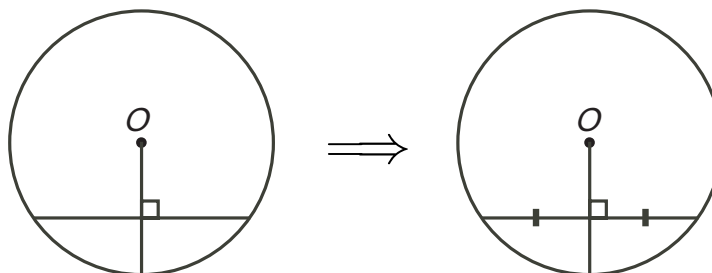


Chord Right Bisector Property

- The right bisector of a chord passes through the centre of the circle.

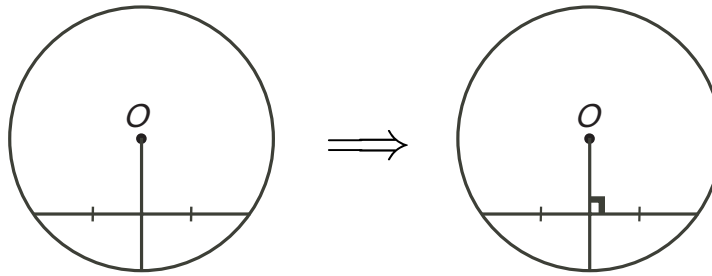


- The perpendicular from the centre to a chord bisects the chord.

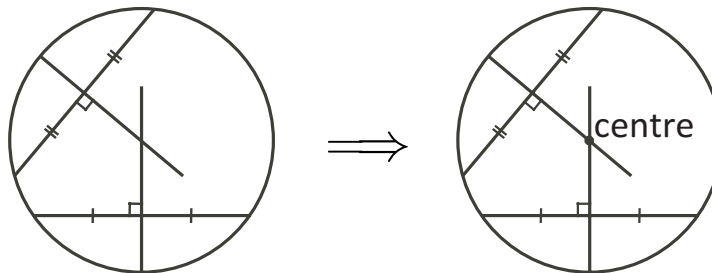




- The line joining the centre to the midpoint of a chord is perpendicular to the chord.



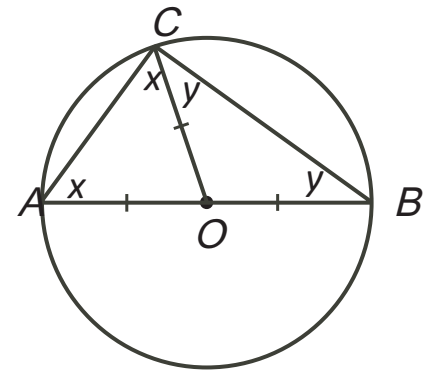
- The centre of a circle is the intersection of the right bisectors of two non-parallel chords.



Prove: The angle inscribed in a semi-circle is 90° .

Proof:

Draw a circle with centre O , diameter AB , and C another point on the circumference of the circle. Join O to C .



In $\triangle OAC$, both OA and OC are radii of the circle. Therefore, $\angle OAC = \angle OCA = x$.

In $\triangle OBC$, both OB and OC are radii of the circle. Therefore, $\angle OBC = \angle OCB = y$.

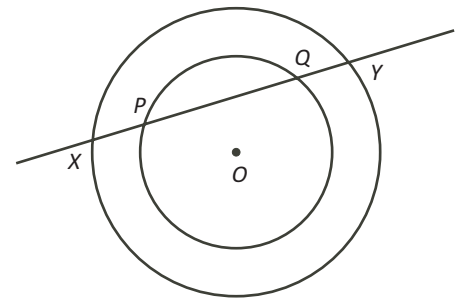
Since the angles in a triangle sum to 180° ,

$$\begin{aligned} \angle BAC + \angle BCA + \angle ABC &= 180^\circ \\ x + (x + y) + y &= 180^\circ \\ 2x + 2y &= 180^\circ \\ x + y &= 90^\circ \end{aligned}$$

But $\angle ACB = x + y$. Therefore, $\angle ACB$, the angle inscribed in the semi-circle, is 90° .

Example 1:

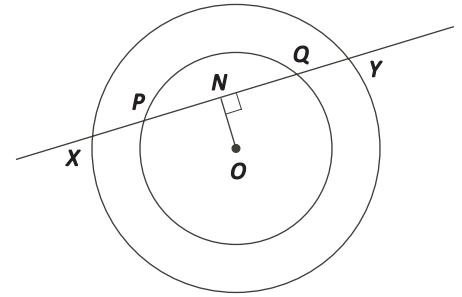
Two circles with the same centre are called *concentric circles*. A line is drawn through two concentric circles intersects the circles at X , P , Q , and Y , as shown. O is the centre of both circles.



Prove that $PX = QY$.

Proof:

Construct a perpendicular from O to the line containing X , P , Q , Y , intersecting the line at N .



PQ is a chord of the smaller circle. ON is a perpendicular to PQ and passes through the centre. Therefore, $PN = NQ$. (1)

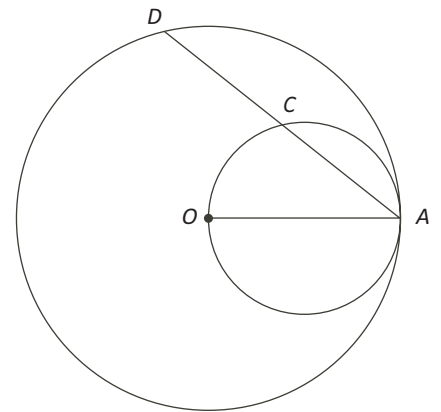
XY is a chord of the larger circle. ON is a perpendicular to XY and passes through the centre. Therefore, $XN = NY$. (2)

$$(2) - (1) \quad XN - PN = NY - NQ$$

$$PX = QY, \text{ follows.}$$

Example 2:

In the diagram, the two circles are tangent at A . If AO is the diameter of the smaller circle and the radius of the larger circle, prove that $AC = CD$.



Proof:

Join OC . $\angle OCA$ is inscribed in a semi-circle since OA is a diameter of the smaller circle. Therefore, $\angle OCA = 90^\circ$.

But O is the centre of the larger circle with $OC \perp AD$. Therefore, C bisects chord AD and $AC = CD$ follows.

