



Intermediate Math Circles

Wednesday October 19 2016

Problem Set 3

N.B. Unless otherwise stated, any point labelled O is assumed to represent the centre of the circle.

1. Determine the length of the chord AB if $OA = 5$ and $ON = 3$.

Solution

In the diagram, the radius $OA = 5$ and $ON = 3$.

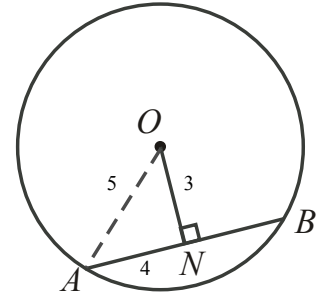
This forms a right triangle $\triangle ONA$. By the Pythagorean Theorem,

$$\begin{aligned} ON^2 + AN^2 &= OA^2 \\ AN^2 &= OA^2 - ON^2 \\ &= 5^2 - 3^2 \\ &= 16 \end{aligned}$$

$$AN = 4 \quad (AN > 0)$$

By the right bisector chord property, N is the midpoint of AB .

So $AN = NB$ and $AB = AN + NB = 4 + 4 = 8$ follows.



2. If $AB = 10$ and $OA = 13$, determine the length of ON .

Solution

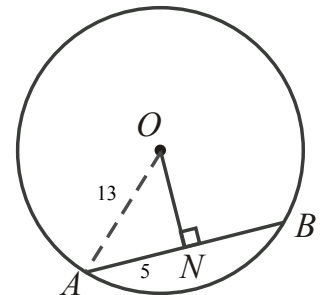
By the right bisector chord property, ON bisects AB .

Hence $AN = NB$. Since $AB = 10$, then $AN = 5$.

Observe $\triangle ONA$ is right angled. By the Pythagorean Theorem,

$$\begin{aligned} ON^2 + AN^2 &= OA^2 \\ ON^2 &= OA^2 - AN^2 \\ ON^2 &= 13^2 - 5^2 \\ &= 144 \end{aligned}$$

$$\therefore ON = 12 \quad (ON > 0)$$



3. A circle has a diameter of length 26. If a chord of the same circle has a length of 10, how far is the chord from the centre?

Solution

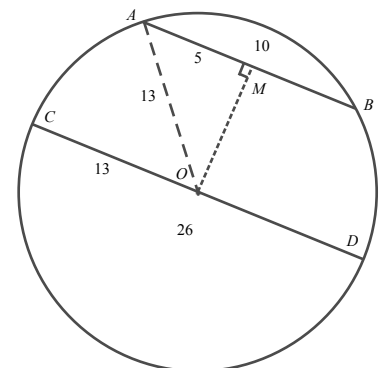
Given chord AB , draw diameter $CD \parallel AB$.

Then the distance from AB to O will be the length of the perpendicular bisector of AB (the perpendicular bisector of a chord passes through the centre of a circle).

Let MO be the perpendicular bisector of AB .

M is therefore the midpoint of AB . Then $AM = BM = 5$.

Note OA is a radius of the circle. Since the circle has diameter 26, the radius $OA = 13$.



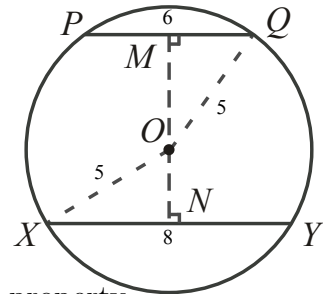


$MO \perp AB$ by construction, and hence $MO \perp AM$. Then $\triangle AMO$ is right angled. By the Pythagorean Theorem,

$$\begin{aligned} MO^2 + AM^2 &= OA^2 \\ MO^2 + 5^2 &= 13^2 \\ MO^2 &= 144 \\ MO &= 12 \quad (MO > 0) \end{aligned}$$

Therefore the distance from the chord to the centre of the circle is 12 units.

4. Calculate the distance between the parallel chords PQ and XY if $PQ = 6$, $XY = 8$, and the radius of the circle is 5.



Solution

The distance between parallel chords is the length of a perpendicular line segment which extends from one to the other.

Let MN be the perpendicular bisector of PQ . By the right bisector property, this passes through O , and it is also the perpendicular bisector of XY .

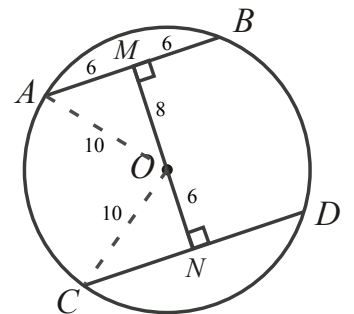
Then $PM = MQ = 3$, $NX = NY = 4$ and the length of MN will be the distance between the chords. Observe OX and OQ are radii. Therefore $OX = OQ = 5$.

Observe that $\triangle MOQ$ and $\triangle ONX$ are right angled. Applying Pythagorean Theorem to both

$$\begin{aligned} OM^2 + MQ^2 &= OQ^2 & ON^2 + NX^2 &= OX^2 \\ OM^2 + 3^2 &= 5^2 & ON^2 + 4^2 &= 5^2 \\ OM^2 &= 16 & ON^2 &= 9 \\ OM &= 4 \quad (OM > 0) & ON &= 3 \quad (ON > 0) \end{aligned}$$

Therefore, $MN = MO + ON = 7$, and so the distance between the chords is 7 units.

5. The two parallel chords AB and CD are a distance of 14 apart. If AB has length 12 and the radius of the circle is 10, calculate the length of CD .



Solution

Let MN be the perpendicular bisector of AB , and hence the perpendicular bisector of CD . The length of MN is the distance between the chords, so $MN = 14$.

If $AB = 12$, then $AM = BM = 6$. Observe OA and OC are radii. Thus $OA = OC = 10$. Apply the Pythagorean Theorem to $\triangle OMA$.

$$\begin{aligned} OM^2 + AM^2 &= OA^2 \\ OM^2 + 6^2 &= 10^2 \\ OM^2 &= 64 \\ OM &= 8 \quad (OM > 0) \end{aligned}$$

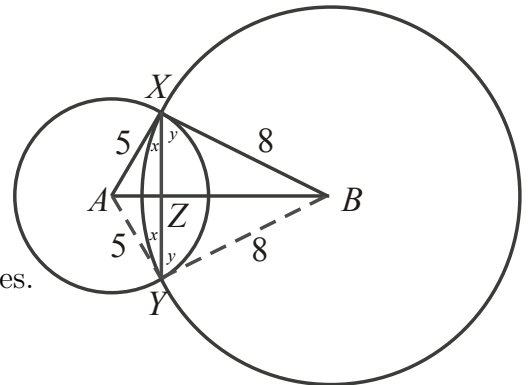


Since $14 = MN = OM + ON = 8 + ON$, then $ON = 6$. Apply the Pythagorean Theorem to $\triangle ONC$

$$\begin{aligned} ON^2 + NC^2 &= OC^2 \\ 6^2 + NC^2 &= 10^2 \\ NC^2 &= 64 \\ NC &= 8 \quad (NC > 0) \end{aligned}$$

So $NC = 8$. But N is the midpoint of CD , so $CN = DN$ and therefore $CD = CN + ND = 16$.

6. Two circles with centre A and B have radii 5 and 8, respectively. The circles intersect at the points X and Y . If $XY = 8$, determine the length of AB , the distance between the centres.



Solution

In the diagram, radii AY and BY are drawn as dotted lines. Let Z mark the intersection of AB and XY .

Since AX and AY are radii of the same circle, $AX = AY$ and hence $\triangle AXY$ is isosceles. So $\angle AXZ = \angle AYX = x$. Similarly, since $BX = BY$, $\triangle BXY$ is isosceles and $\angle BXZ = \angle BYX = y$.

Observe $\angle AXB = x + y = \angle AYB$. Then by SAS, $\triangle AXB \cong \triangle AYB$.

It follows that, $\angle XAB = \angle YAB$. By ASA, $\triangle AXZ \cong \triangle AYZ$ ($AX = AY, \angle AXZ = \angle AYZ, \angle XAB = \angle YAB$) and hence $\angle AZX = \angle AZY$.

But XY is a straight line, so $\angle AZX = \angle AZY = 90^\circ$, and similarly $\angle BZX = \angle BZY = 90^\circ$.

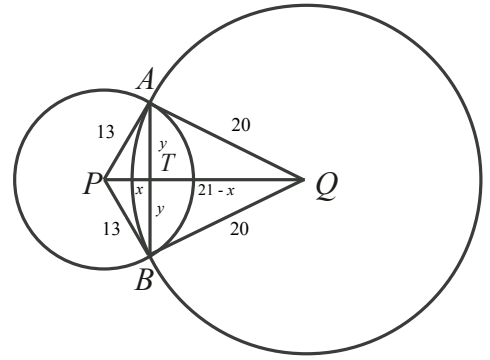
Thus $\triangle XZA$ and $\triangle XZB$ are right angled. Furthermore, $XZ = ZY = 4$ since $XY = 8$. Applying the Pythagorean Theorem to the two triangles gives

$$\begin{aligned} XZ^2 + AZ^2 &= XA^2 & XZ^2 + ZB^2 &= XB^2 \\ 4^2 + AZ^2 &= 5^2 & 4^2 + ZB^2 &= 8^2 \\ AZ^2 &= 9 & ZB^2 &= 48 \\ AZ &= 3 \quad (AZ > 0) & ZB &= \sqrt{48} \quad (ON > 0) \end{aligned}$$

Therefore the distance between the centres of the circles is $AB = AZ + ZB = 3 + \sqrt{48}$ (Note: $\sqrt{48}$ can be simplified to $3 + 4\sqrt{3}$ so $AB = 3 + 4\sqrt{3}$).



7. In the diagram, $PA = 13$ and $QA = 20$, where P and Q are the centres of the circles. Determine the length of AB if $PQ = 21$.



Solution

Observe that this problem is similar to Problem 6.

Thus, using the same reasoning, $AT = TB = y$,

$\angle ATP = \angle BTP = \angle ATQ = \angle BTQ = 90^\circ$,

and $\triangle ATP$, $\triangle ATQ$ are right angled.

Let $x = PT$. Since $PQ = 21$ and $PT = x$, then $QT = 21 - x$. By the Pythagorean Theorem:

$$PT^2 + AT^2 = AP^2$$

$$x^2 + y^2 = 13^2 \quad (1)$$

$$QT^2 + AT^2 = AQ^2$$

$$(21 - x)^2 + y^2 = 20^2$$

$$441 - 42x + x^2 + y^2 = 20^2 \quad (2)$$

From (1), substitute 13^2 for $x^2 + y^2$ in (2) to get

$$441 - 42x + 13^2 = 20^2$$

$$441 - 42x = 231$$

$$-42x = 231 - 441$$

$$-42x = -210$$

$$x = 5$$

Substitute $x = 5$ into (1) to solve for y

$$x^2 + y^2 = 13^2$$

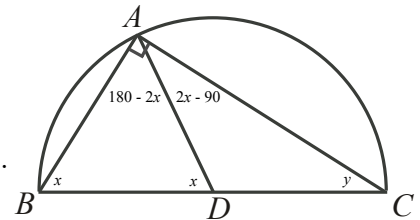
$$y^2 = 13^2 - 5^2$$

$$y^2 = 144$$

$$y = 12 \quad (y > 0)$$

Therefore $AB = AT + TB = 2y = 24$.

8. In the diagram, $\triangle ABC$ is inscribed in the semicircle with centre D . If $AB = AD$, determine the measure of $\angle ACD$.



Solution

Any angle inscribed in a semi-circle is right angled, so $\angle BAC = 90^\circ$.

Since $AB = AD$, $\triangle ABD$ is isosceles and $\angle ABD = \angle ADB = x$.

Then $\angle BAD = 180 - 2x$, and $\angle DAC = 90 - \angle BAD = 2x - 90$.

AD and DC are radii, so $AD = DC$ and thus $\triangle ADC$ is isosceles. Therefore $\angle ACD = \angle CAD$ (1) and $y = 2x - 90$ (2) follows.

Setting (1) and (2) equal to each other gives $2x - 90 = 90 - x$. Then $3x = 180$ and $x = 60^\circ$ follows.

Substituting $x = 60$ into (1), $y = 2(60) - 90 = 30$. Therefore, $\angle ACD = 30^\circ$.



Alternate Solution

Since DA and DB are radii, $AD = DB$. But $AD = DB$. Therefore, $AD = DB = AB$ and $\triangle ABD$ is equilateral. It follows that each angle in $\triangle ABD$ is 60° .

Since DA and DC are radii, $\triangle ADC$ is isosceles and $\angle CAD = \angle DCA = y$.

Observe that $\angle ADB$ is exterior to $\triangle ADC$. Then $60^\circ = \angle ADB = \angle DAC + \angle DCA = y + y$. So $2y = 60^\circ$ and hence $y = 30^\circ$.

9. In the diagram, $\triangle XYZ$ is right-angled at Z . W is the midpoint of XY , and the circle with diameter ZW intersects WX at V . If $XY = 50$ and $WV = 7$, determine the length of XZ .

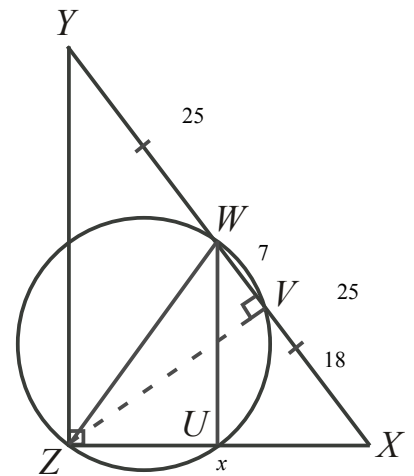
Solution

In the diagram, since W is the midpoint of XY , $WY = WX = 25$. Furthermore, if $WV = 7$, then $VX = 18$.

Because V lies on the circle and ZW is a diameter, $\angle WVZ = \angle ZVX = 90^\circ$ (angle inscribed in a semi-circle).

Consider $\triangle WUX$. Since U is on the circumference, $\angle WUZ = \angle WUX = 90^\circ$. Then $\triangle WUX \sim \triangle YZX$ by AAA.

So $\frac{WX}{YX} = \frac{25}{50} = \frac{1}{2} = \frac{XZ}{UZ}$. Hence $XZ = 2UZ$ so $UX = UZ$.



But then $\triangle WUX \cong \triangle WUZ$ by SAS since they share common side WU . So $ZW = XW = 25$.

Now, $\triangle WVZ$, $\triangle ZVX$ are right-angled. By the Pythagorean Theorem,

$$\begin{aligned} WV^2 + ZV^2 &= ZW^2 \\ 7^2 + ZV^2 &= 25^2 \\ ZV^2 &= 25^2 - 7^2 \\ ZV^2 &= 576 \end{aligned}$$

and

$$\begin{aligned} ZX^2 &= VX^2 + ZV^2 \\ ZX^2 &= 18^2 + 576 \\ ZX^2 &= 900 \\ \therefore ZX &= 30 \quad (ZX > 0) \end{aligned}$$



Alternate Solution

In the previous solution, it was shown that $\triangle VZX$ was right angled. But $\triangle VZX$ also shares common angle $\angle YXZ$ with $\triangle ZYX$.

Hence $\triangle VZX \sim \triangle ZYX$. Let $x = ZX$ as shown. Using the properties of similar triangles,

$$\begin{aligned}\frac{VX}{ZX} &= \frac{ZX}{YX} \\ \frac{VX}{x} &= \frac{x}{YX} \\ VX \cdot YX &= x^2 \\ 18 \cdot 50 &= x^2 \\ 900 &= x^2 \\ 30 &= x \quad (x > 0)\end{aligned}$$

Hence $x = ZX = 30$.