Intermediate Math Circles
Wednesday October 26, 2016
Mathematical Games

A combinatorial game is a game played between two players where there is no chance involved. More precisely, a game where both players can see everything that is going on, the players move alternately, and both players have the same available moves.
One neat thing about combinatorial games is that exactly one of the two players always has a **winning strategy**. To have a winning strategy means that no matter what moves their opponent makes, they can counter with a sequence of moves that will eventually cause them to win the game. A winning strategy need not be easy to find, but it always exists.

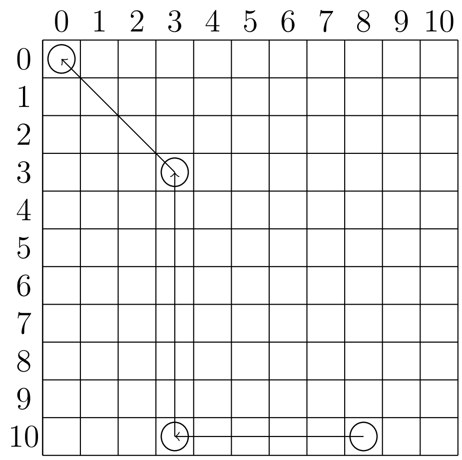
Game 1 (Fifteen)**.** In fifteen, two players take turns removing either 1 or 2 stones from a pile which initially contains 15. The player that removes the last stone wins. Here is an example of a game:

- Player 1 removes 2 stones, leaving 13.
- Player 2 removes 1 stone, leaving 12.
- Player 1 removes 2 stones, leaving 10.
- Player 2 removes 2 stones, leaving 8.
- Player 1 removes 1 stone, leaving 7.
- Player 2 removes 2 stones, leaving 5.
- Player 1 removes 2 stones, leaving 3.
- Player 2 removes 1 stone, leaving 2.
- Player 1 removes 2 stones and wins.

Game 2 (Nim)**.** In Nim, there are several piles of several stones. On their turn, a player can remove as many stones as they like from one pile. The players alternate removing stones according to this rule, and the player to take the last stone wins. We use the notation $n_1 \oplus n_2 \oplus \cdots \oplus n_k$ to denote the game that starts with k piles, having n_1, $n_2, \ldots,$ and n_k stones. For example, $1 \oplus 2 \oplus 3$ refers to the situation where there are three piles having one stone, two stones, and three stones. $8 \oplus 10$ refers to the game starting with a pile of eight stones and a pile of ten stones. If there is only one pile, we just denote the game by that integer. For example, 7 denotes the game with one pile of 7 stones. Here is an example of a game starting at $8 \oplus 10$:

- Player 1 removes 6 stones from the pile with 8 leaving $2 \oplus 10$.
- Player 2 removes 1 stone form the plie of 10 leaving $2 \oplus 9$.
- Player 1 removes all 9 stones from the pile of 9 leaving 2.
- Player 2 removes the remaining 2 stones and wins.

Figure 1: A Left Handed Queen example



Game 3 (The Left Handed Queen)**.** This game is played on a rectangular grid using a queen from chess (or a coin, or a thumb tack[1]). The grid need not be an 8×8 grid. The queen starts in some cell (other than the top left) and two players alternate moving the queen any distance horizontally left, vertically up, or diagonally up-left. The player who puts the queen in the top left cell wins. Figure 1 has a sequence of legal moves starting at $(8, 10)$. In the example, Player 1 wins.

Game 4 (Wythoff's game)**.** This game is similar to Nim. There are two piles of stones. Players alternate removing stones, and, as usual, the player to remove the last stone wins. This time, there are two types of legal move: A player may remove any number of stones from one pile, or the same number of stones from each pile. We will use the same notation as Nim. For example, if the game starts at $5 \oplus 10$, the play might go as follows:

- Player 1 removes 3 stones from each pile leaving $2 \oplus 7$.
- Player 2 removes 5 stones from the pile of 7, leaving $2 \oplus 2$.
- Player 1 removes one stone from each pile leaving $1 \oplus 1$.
- Player 2 removes one stone from each pile and wins.

[1] not recommended

Game 5 (Fibonacci Nim)**.** This game is another variation on Nim. There is one pile of stones. The first player to move can remove any number of stones as long as they take at least one and leave at least one. From then on, each player can remove no more than twice the number of stones that their opponent removed on the turn immediately before. The player to remove the last stone wins. Here is an example of a game starting with 20 stones:

- Player 1 removes 3 stones, leaving 17.
- Player 2 removes 5 stones, leaving 12.
- Player 1 removes 2 stones, leaving 10.
- Player 2 removes 4 stones, leaving 6.
- Player 1 removes the rest of the stones and wins.