Intermediate Math Circles
Wednesday October 26, 2016
Problem Set 4

Here are the October 26 exercises. The last three are pretty tricky. Have fun!

Game 1 (Fifteen)

1. Find a partner and play Fifteen for 5-7 minutes. Does Player 1 or Player 2 have a winning strategy? What is the strategy?

2. A simple variation of Fifteen has the same rules except the pile starts with a different number of stones. Which player has a winning strategy if there are 20 stones? What if there are 30 stones? What about 1000 stones?

3. Given any positive integer, n, how can you tell which player has a winning strategy if the pile starts with n stones? What is the strategy?

4. Change the rules again so that players may remove one, two, or three stones on their turn. How do the strategies change in Exercises 2 and 3?

Game 2 (Nim)

5. This and the next problem are about Nim.

   (a) Which player has a winning strategy if there is a single pile?

   (b) Which player has a winning strategy for $2 \oplus 2$?

   (c) Which player has a winning strategy for $2 \oplus 3$?

   (d) Which player has a winning strategy for $4 \oplus 4$?

   (e) Which player has a winning strategy if there are two equal piles?

   (f) Which player has a winning strategy if there are two piles which are not equal?

   (g) Describe the winning strategy for an arbitrary game with two piles [Hint: you have really already done this if you did (e) and (f).]

6. Using your answers from Exercise 4 (e) and (f), draw partial game trees for the following:

   (a) $1 \oplus 1 \oplus 1$

   (b) $1 \oplus 1 \oplus 2$

   (c) $2 \oplus 2 \oplus 1$

   (d) $1 \oplus 2 \oplus 3$.

   You may use earlier parts to make later parts easier. Use your game tree to extract a strategy to beat a family member later.

7. See if you can figure out which player has a winning strategy (and what the strategy is) for Nim in some other general situations. For example,
   (a) Every pile has one stone (any number of piles).
   (b) Every pile has two stones (any number of piles).
   (c) Any number of piles, but every pile has either one stone or two stones.

   The winning strategies of Nim are completely known, but somewhat complicated in general. It will be described in the online notes.

Game 3 (The Left Handed Queen)

8. Play the left handed queen game with a friend (or by yourself) for ten minutes. Try to identify which player has a winning strategy for various starting positions.
9. If you leave the queen in $(1, 2)$, your opponent can not win on their next turn. Furthermore, you are guaranteed to be able to win on your next move. Convince yourself of this and find more cells that leave you with a winning strategy if you can put the queen there. Try to identify a pattern.

Game 4 (Wythoff's game)

10. Play Wythoff's game for a few minutes and try to identify which player has a winning strategy in a few small cases. For example, $1 \oplus 2$, $2 \oplus 5$, $3 \oplus 4$, etc.
11. Explain how you can use a strategy from the left handed queen in Wythoff's game.

Game 5 (Fibonacci Nim)

12. Play Fibonacci Nim for a few minutes. Which player has a winning strategy when there are 5, 10, or 20 stones. What is the strategy. Warning: This one is hard. The strategy involves the Zackendorf decomposition of the number.
13. Find the Zackendorf decompositions of 10, 15, 20, 500, 610, and 1000.
14. Explain why no two numbers in a Zackendorf decomposition are consecutive Fibonacci numbers. Here consecutive means they appear next to one another in the Fibonacci sequence.
15. Explain why the same number can not occur twice in the Zackendorf decomposition of a number.
16. Find a general strategy for Nim.