Intermediate Math Circles
Wednesday October 26 2016
Problem Set 4

Here are the October 26 exercises. The last three are pretty tricky. Have fun!

Game 1 (Fifteen)

1. Find a partner and play Fifteen for 5-7 minutes. Does Player 1 or Player 2 have a winning strategy? What is the strategy?

   Solution

   Player 2 has a winning strategy. The strategy is to always do the "opposite" move that Player 1 just did. That is, if Player 1 takes 2, Player 2 should take 1, and if Player 1 takes 1, Player 2 should take 2. This ensures that Player 2 always leaves Player 1 with a multiple of 3. If Player 1 is always left with smaller and smaller multiples of 3, they must eventually be left with 0, which means Player 2 took the last stone.

2. A simple variation of Fifteen has the same rules except the pile starts with a different number of stones. Which player has a winning strategy if there are 20 stones? What if there are 30 stones? What about 1000 stones?

   Solution

   With problem 1 in mind, the goal is always to leave your opponent with a multiple of 3. In this case, Player 1 can take 2 immediately and leave Player 2 with 18 stones, which is a multiple of 3. Using the "opposite move" strategy from the previous problem, Player 1 can win no matter what Player 2 does. When there are 30 stones, Player 2 again has a winning strategy because 30 is a multiple of 3. No matter what Player 1 does, Player 2 can counter with the opposite move and leave Player 1 with 27 stones, then 24 stones, and so on. Since 1000 is not a multiple of 3, Player 1 can take 1 on their first turn and leave Player 2 with 999 stones, which is a multiple of 3. The strategy is the same.

3. Given any positive integer, n, how can you tell which player has a winning strategy if the pile starts with n stones? What is the strategy?

   Solution

   In general, If n is a multiple of 3, Player 2 has a winning strategy, and Player 1 has a winning strategy has a winning strategy otherwise. If n is a multiple of 3, Player 2's winning strategy is to always counter with the opposite move that Player 1 makes. If n is not a multiple of 3, Player 1 should, on their first move, leave Player 2 with a multiple of 3. Whether they need to take 1 or 2 depends on the number. After that, Player 1 should always make the opposite move that Player 2 just made.

4. Change the rules again so that players may remove one, two, or three stones on their turn. How do the strategies change in Exercises 2 and 3?

Solution

The strategy only changes a bit. Imagine if Player 1 was left with 4 stones. Then, no matter what move they made, Player 2 could win on their next turn. Similarly, if Player 1 was left with 8 stones, no matter what they do, Player 2 can counter with a move that leaves Player 1 with 4 stones. Since Player 1 loses if they are left with 4 stones, they also lose if they are left with 8 stones. Therefore, like the multiples of 3 were important in the previous game, multiples of 4 are important in this modified game. The strategy is as follows:

- If n is a multiple of 4, Player 2 has a winning strategy. The strategy is to always remove enough stones to leave Player 1 with a multiple of 4.
- If n is not a multiple of 4, Player 1 has a winning strategy. The strategy is the same as Player 2's strategy from the previous case, except for the first move which should be to leave Player 2 with a multiple of 4. Since n is not a multiple of 4, it is either 1, 2, or 3 more than a multiple of 4, so this can be done.

Game 2 (Nim)

5. This and the next problem are about Nim.

(a) Which player has a winning strategy if there is a single pile?

(b) Which player has a winning strategy for $2 \oplus 2$?

(c) Which player has a winning strategy for $2 \oplus 3$?

(d) Which player has a winning strategy for $4 \oplus 4$?

(e) Which player has a winning strategy if there are two equal piles?

(f) Which player has a winning strategy if there are two piles which are not equal?

(g) Describe the winning strategy for an arbitrary game with two piles [Hint: you have really already done this if you did (e) and (f).]

Solution

(a) If there is a single pile, Player 1 has a winning strategy. The strategy is to take all of the stones.

(e) If there are two equal piles, Player 2 has a winning strategy. The strategy is to always do the same thing that Player 1 does in order to keep leaving Player 1 with two equal piles. Since, on every move, players must remove at least one stone, Player 1 will keep getting left with two smaller and smaller equal piles. This process can't go on forever, so eventually, they will be left with two piles of 0. The only way this can happen is if Player 2 took the last stone and won.

(f) Player 1 has a winning strategy with two unequal piles. The strategy is to first remove enough stones from the larger pile to leave Player 2 with two equal piles, and then follow the strategy from (e). That is, always leave Player 2 with two equal piles.

(g) (e)+(f)

Figure 1: Game tree for $1 \oplus 1 \oplus 1$

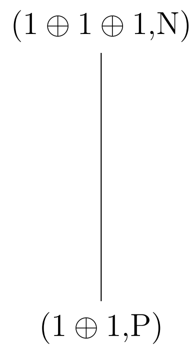
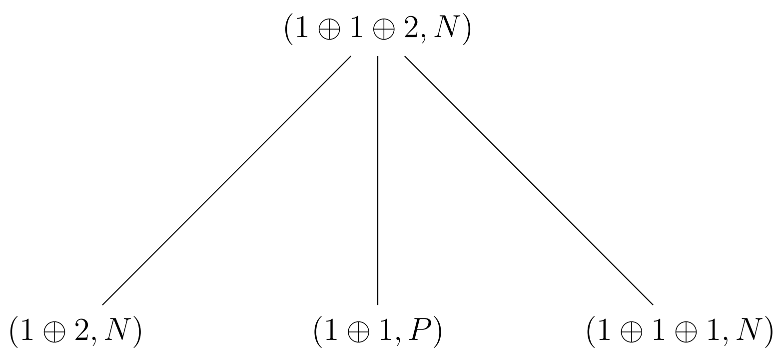


Figure 2: Game tree for $1 \oplus 1 \oplus 2$



6. Using your answers from Exercise 4 (e) and (f), draw partial game trees for the following:

   (a) $1 \oplus 1 \oplus 1$

   (b) $1 \oplus 1 \oplus 2$

   (c) $2 \oplus 2 \oplus 1$

   (d) $1 \oplus 2 \oplus 3$.

   You may use earlier parts to make later parts easier. Use your game tree to extract a strategy to beat a family member later.

   Solution

   (a) See Figure 1

   (b) See Figure 2

   (c) See Figure 3. Note that $1 \oplus 2 \oplus 1$ is the same as $1 \oplus 1 \oplus 2$.

   (d) See Figure 4

Figure 3: Game tree for $2 \oplus 2 \oplus 1$

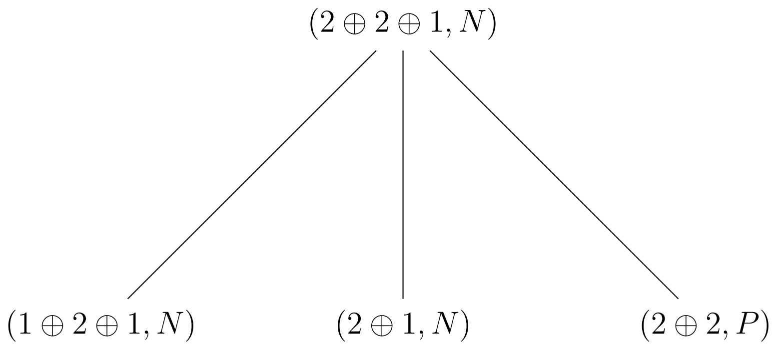
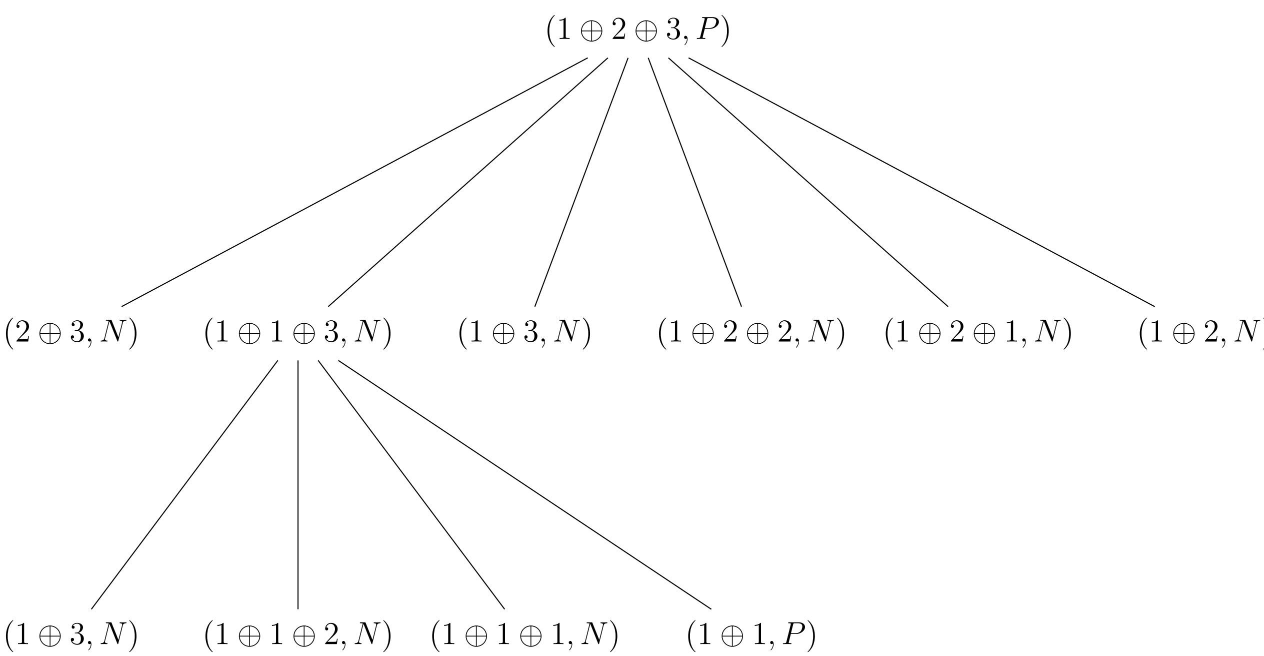


Figure 4: Game tree for $1 \oplus 2 \oplus 3$



7. See if you can figure out which player has a winning strategy (and what the strategy is) for Nim in some other general situations. For example,

   (a) Every pile has one stone (any number of piles).

   (b) Every pile has two stones (any number of piles).

   (c) Any number of piles, but every pile has either one stone or two stones.

   The winning strategies of Nim are completely known, but somewhat complicated in general. It will be described in the online notes.

   Solution

   (a) If every pile has one stone, neither player ever has a choice. If there is an odd number of stones, Player 1 will win, and if there is an even number of stones, Player 2 will win.

   (b) We will look at two cases. First, if there is an even number of piles, Player 2 can guarantee a win by always doing the same thing Player 1 did. That is, if Player 1 takes a whole pile of 2, Player 2 should take a whole pile of 2. If Player 1 takes one stone from a pile of 2, Player 2 should take one stone from a pile of 2. If Player 1 removes a whole pile of 1, Player 2 should remove a whole pile of 1. If Player 2 does this, There will always be an even number of piles of one stone and an even number of piles of two stones. This means, no matter what Player 1 does, Player 2 always has the same move available. If Player 1 had a pile of 2 stones available, Player 2 will also have a pile of two stones available. This is true for the following reason: Player 1 had a pile of two stones available, and when it is Player 1's turn, there is an even number of piles of two stones, so this even number is at least 1. An even number that is at least 1 is at least 2. Therefore, Player 2 has a pile of two stones available. This sort of reasoning works in other cases. No matter what move Player 1 made, Player 2 will have that same move available.

       In the case that there are an odd number of piles, Player 1 has a winning strategy. It is to take a whole pile and then follow the previous "copying" strategy for the case with an even number of piles.

(c) There are 4 cases:

- An even number of piles of one stone and an even number of piles of two stones: In this case, Player 2 has a winning strategy which is to copy the moves of Player 1 as in the previous exercise.
- An even number of piles of one stone and an odd number of piles of two stones: In this case, Player 1 has a winning strategy which is to take a whole pile of two stones. This leaves Player 2 in the situation where there is an even number of piles of one stone and an even number of piles of two stones. Player 1, from then on, uses the copying strategy.
- An odd number of piles of one stone and an even number of piles of two stones: Similar to the previous, Player 1 should take a whole pile of one stone, leaving Player 2 with an even number of piles of one stone and an even number of piles of two stones.
- An odd number of piles of one stone and an odd number of piles of two stones: In this case, Player 1 should take one stone from a pile of two. This will increase the number of piles of one stone by 1, and decrease the number of piles of two stones by 1. Hence, there will be an even number of both, so Player 1 can continue the copying strategy to win as in the previous two cases.

Game 3 (The Left Handed Queen)

8. Play the left handed queen game with a friend (or by yourself) for ten minutes. Try to identify which player has a winning strategy for various starting positions.

9. If you leave the queen in $(1, 2)$, your opponent can not win on their next turn. Furthermore, you are guaranteed to be able to win on your next move. Convince yourself of this and find more cells that leave you with a winning strategy if you can put the queen there. Try to identify a pattern.

Solution

Using the principles from game trees, you can label the cells by **N** or **P**. **N** means the next player to move will win. Therefore, the left column, the top row, and the diagonal should all be **N** because the next player can win immediately. Here is the start of our labelling. We will number the columns and rows starting with 0:

	0	1	2	3	4	5	6	7	8	9	10
0		**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**
1	**N**	**N**									
2	**N**		**N**								
3	**N**			**N**							
4	**N**				**N**						
5	**N**					**N**					
6	**N**						**N**				
7	**N**							**N**			
8	**N**								**N**		
9	**N**									**N**	
10	**N**										**N**

If there is no choice but to move to a cell labelled by **N**, then no matter what the next player does, their opponent will win. This means it is a previous player win. The cells $(1, 2)$ and $(2, 1)$ have this property, so we can label them by **P** and if you look at it, if it is your move and the queen is in either of these two cells, you are done for. Here is the picture:

	0	1	2	3	4	5	6	7	8	9	10
0		**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**
1	**N**	**N**	**P**								
2	**N**	**P**	**N**								
3	**N**			**N**							
4	**N**				**N**						
5	**N**					**N**					
6	**N**						**N**				
7	**N**							**N**			
8	**N**								**N**		
9	**N**									**N**	
10	**N**										**N**

If a player can move the queen to a cell labelled with a **P**, they have a winning strategy. This is, again, because immediately after a player moves, they become the previous player. Therefore, any cell that has "access" to a cell labelled by **P** should be labelled by **N**:

	0	1	2	3	4	5	6	7	8	9	10
0		**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**
1	**N**	**N**	**P**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**
2	**N**	**P**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**
3	**N**	**N**	**N**	**N**	**N**						
4	**N**	**N**	**N**	**N**	**N**	**N**					
5	**N**	**N**	**N**		**N**	**N**	**N**				
6	**N**	**N**	**N**			**N**	**N**	**N**			
7	**N**	**N**	**N**				**N**	**N**	**N**		
8	**N**	**N**	**N**					**N**	**N**	**N**	
9	**N**	**N**	**N**						**N**	**N**	**N**
10	**N**	**N**	**N**							**N**	**N**

The cells $(3, 5)$ and $(5, 3)$ have now appeared as cells that have no choice but to move to a cell labelled by **N**, so they must be labelled by **P**. If you continue this reasoning, you will end up with the following:

	0	1	2	3	4	5	6	7	8	9	10
0		**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**
1	**N**	**N**	**P**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**
2	**N**	**P**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**
3	**N**	**N**	**N**	**N**	**N**	**P**	**N**	**N**	**N**	**N**	**N**
4	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**P**	**N**	**N**	**N**
5	**N**	**N**	**N**	**P**	**N**	**N**	**N**	**N**	**N**	**N**	**N**
6	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**P**
7	**N**	**N**	**N**	**N**	**P**	**N**	**N**	**N**	**N**	**N**	**N**
8	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**
9	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**
10	**N**	**N**	**N**	**N**	**N**	**N**	**P**	**N**	**N**	**N**	**N**

It seems like almost all of the cells are going to be labelled by **N**, so understanding the game really seems to be about understanding where the **P** cells arise. Here is a list of the ones we have so far, and a few more that you would find if you continued with a bigger grid:

$(1, 2)$	$(2, 1)$
$(3, 5)$	$(5, 3)$
$(4, 7)$	$(7, 4)$
$(6, 10)$	$(10, 6)$
$(8, 13)$	$(13, 8)$
$(9, 15)$	$(15, 9)$
$(11, 18)$	$(18, 11)$
$(12, 21)$	$(20, 12)$
$(14, 23)$	$(23, 14)$
$(16, 26)$	$(26, 16)$

The pattern can be understood as follows: Imagine that you start to place the positive integers $(1, 2, 3, 4, \dots)$ in boxes according to some rules. To start with, each box has to have exactly two numbers, and each number goes in exactly one box. Also, the first box has to have two numbers that are 1 apart, the second box has to have numbers that are 2 apart, the third box has to have numbers that are three apart, and so on. You start by putting 1 in box 1. According to the rule, 2 also has to go in box 1 because it is the only available number that is 1 away from 1. Now you look at the next box and start it with the smallest unused number, which is 3. So 3 goes in box 2, but according to the rule, the other number in box 2 has to be 2 away from 3, and the only such available number is 5. Now you start the next box with the smallest unused number. So far, we have used 1,2,3, and 5, so the smallest unused number is 4. Therefore, 4 goes in box 3, and the only number available that is 3 away from 4 is 7. This means 7 goes in box 3. So far, box 1 contains 1 and 2, box 2 contains 3 and 5, and box 3 contains 4 and 7. You see that 1,2,3,4,5, and 7 have been used, so you start box 4 with 6, and you are forced to put 10 in box 4 as well. This goes on for as long as you want and the numbers in the boxes give you the pairs. For example, since 14 and 23 end up in a box together, we get two **P** cells, which are $(14, 23)$ and $(23, 14)$. This is exactly how all of the **P** cells arise.

Here is a fact: If the queen is on a **P** cell, the only available moves are to **N**-cells, and if the queen is on an **N** cell, there is an available move to a **P** cell. Therefore, if the queen starts on a **P** cell, Player 2 has a winning strategy, which is to always move the queen to a **P** cell. If the queen starts on an **N** cell, Player 1 has a winning strategy, which is, again, to always move the queen to a **P** cell.

Game 4 (Wythoff's game)

10. Play Wythoff's game for a few minutes and try to identify which player has a winning strategy in a few small cases. For example, $1 \oplus 2$, $2 \oplus 5$, $3 \oplus 4$, etc.

11. Explain how you can use a strategy from the left handed queen in Wythoff's game.

    Solution

    Wythoff's game is, in some sense, exactly the same game as the left handed queen. It is just played by keeping track of the coordinates of the queen rather than using the board. Moves to the left decrease the horizontal coordinate, which is the same as removing stones from the left pile. Vertical moves correspond to decreasing the vertical coordinate, which is the same as removing stones from the right pile. Up-left diagonal moves correspond to decreasing the horizontal and vertical coordinates equally, which is the same as removing the same number of stones from each pile. The strategy is the same. For the game $n \oplus m$, if n and m appear in a box together (see the solution to 9), then Player 2 has a winning strategy. Otherwise, Player 1 has a winning strategy. The strategy in either case is to always make a move that leaves your opponent with two piles, whose sizes occur in a box together.

Game 5 (Fibonacci Nim) (Solutions coming after week 5!)

12. Play Fibonacci Nim for a few minutes. Which player has a winning strategy when there are 5, 10, or 20 stones. What is the strategy. Warning: This one is hard. The strategy involves the Zackendorf decomposition of the number.

13. Find the Zackendorf decompositions of 10, 15, 20, 500, 610, and 1000.

14. Explain why no two numbers in a Zackendorf decomposition are consecutive Fibonacci numbers. Here consecutive means they appear next to one another in the Fibonacci sequence.

15. Explain why the same number can not occur twice in the Zackendorf decomposition of a number.

16. Find a general strategy for Nim.