Intermediate Math Circles
October 05, 2016
Geometry I: Angles

Over the next four weeks, we will look at several geometry topics. Some of the topics may be familiar to you while others, for most of you, will not. We will start with angle facts and some simple proofs.

Warming Up

Refer to the video for the solution to the warm-up problem.

Getting Started

In geometry, there are certain terms that we just understand. For example, if you were asked to define an angle, you might struggle with appropriate words but could easily draw an angle. Try writing a definition.

Definition: The space, formed by two lines or rays diverging from a common point (the vertex). The space is commonly measured in degrees.

What follows is a short list of definitions that will be useful in our discussions.

Angle Related Definitions

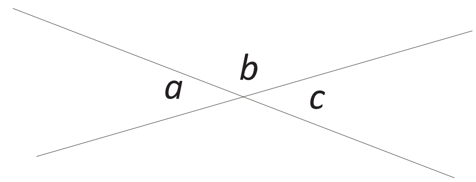
An *acute angle* is any angle measuring between 0° and 90°.
A *right angle* is an angle measuring 90°.
An *obtuse angle* is any angle measuring between 90° and 180°.
A *straight* angle is an angle measuring 180°.

Two angles whose sum is 180° are called *supplementary* angles.
Two angles whose sum is 90° are called *complementary* angles.

A *scalene* triangle has three sides of different length.
An *isosceles* triangle has two sides of equal length.
An *equilateral* triangle has three sides of equal length.

When two lines intersect, four angles are formed. The angles that are directly opposite to each other are called *opposite* angles.

Prove: Opposite angles are equal.



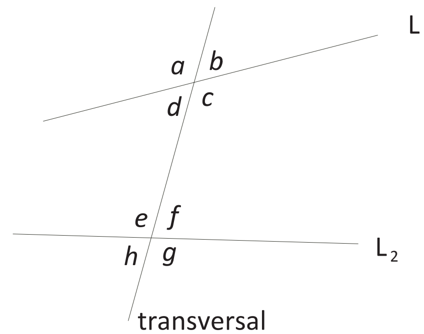
Proof:

We want to show that $a = c$.

Since a and b form a straight angle, $a + b = 180°$. (1)
Since c and b form a straight angle, $c + b = 180°$. (2)

In (1) and (2), since $180° = 180°$, then $a + b = c + b$.
b is common to both sides so $a = c$ follows.

Therefore, opposite angles are equal.



A *transversal* is any line that intersects two (or more) lines at different points.

Two angles located on the same side of the transversal between the two lines are called $co - interior$ angles.

d and e are co-interior angles.
c and f are co-interior angles.

Two non-adjacent angles located on opposite sides of the transversal between the two lines are called *alternate* (*interior*) angles.

d and f are alternate (interior) angles.
c and e are alternate (interior) angles.

Two angles located at the same location relative to where the transversal intersects each line are called *corresponding* angles.

a and e are corresponding angles.
b and f are corresponding angles.
c and g are corresponding angles.
d and h are corresponding angles.

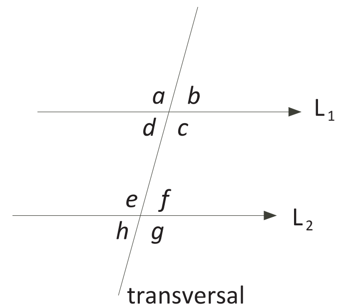
An *axiom* is a logical statement which is assumed to be true. We just accept the truth of the statement.

Two things that are equal to the same thing are equal to each other. If $A = B$ and $A = C$ then $B = C$. Most people accept this logic as true and do not give it a second thought.

We prove statements in geometry using definitions, axioms and (previously) proven facts.

Axiom:

If a transversal cuts two parallel lines, then the co-interior angles are supplementary. That is, the two co-interior angles add to 180°.

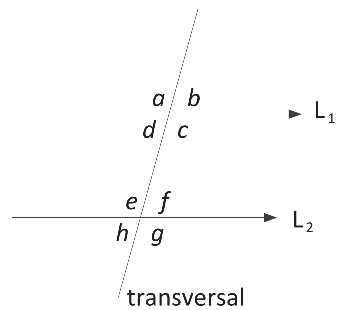


We just accept it to be true. So in this case, $d + e = 180°$ and $c + f = 180°$.

Another axiom that we could have started with is: "the angles in a triangle sum to 180°." As a result of starting with the first axiom, we will be able to prove the second axiom (it will not be an axiom for us).

Prove:

If a transversal cuts two parallel lines, then the alternate angles are equal and the corresponding angles are equal.



Proof:

We will show that two alternate interior angles d and f are equal.

Since d and c form a straight line, $d + c = 180°$. (1)

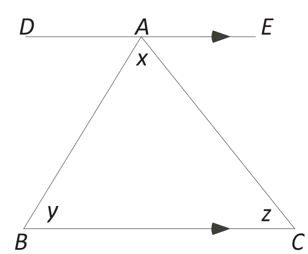
Since $L_1 \parallel L_2$, then using the axiom, $f + c = 180°$. (2)
In (1) and (2), since $180° = 180°$, then $d + c = f + c$.
c is common to both sides so $d = f$ follows.

Therefore, alternate interior angles are equal.

The proof of the equality of the alternate exterior angles and the equality of corresponding angles is very similar to the proof presented here and, thus, will not provided here.

Prove:

In any triangle, the sum of the interior angles is 180°.



Proof:

Construct $\triangle ABC$.
Let $\angle BAC = x$, $\angle ABC = y$, and $\angle ACB = z$.

Through A draw line segment DE parallel to BC.

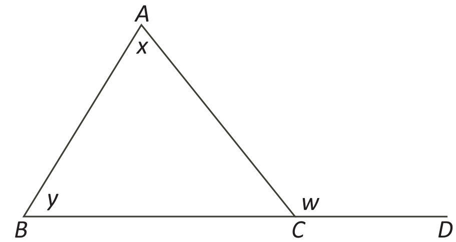
Since $DE \parallel BC$, $\angle DAB = \angle ABC = y$ and $\angle EAC = \angle ACB = z$.

$$\begin{aligned} \angle ABC + \angle BAC + \angle ACB &= y + x + z \\ &= \angle DAB + \angle BAC + \angle EAC \\ &= 180° \qquad \angle DAB,\ \angle BAC,\ \angle EAC \text{ form a straight angle.} \end{aligned}$$

$\therefore$ the angles in a triangle sum to 180°.

Definition:

An exterior (or external) angle is the angle between one side of a triangle and the extension of an adjacent side.



Prove:

An exterior angle of a triangle is equal to the sum of the opposite interior angles.

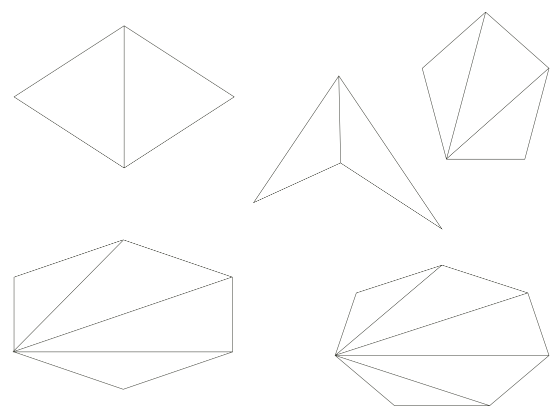
Proof:

Construct $\triangle ABC$ with $\angle BAC = x$, $\angle ABC = y$ and exterior $\angle ACD = w$ as shown.
Since BCD is a straight angle, $\angle BCA + w = 180°$. (1)
In $\triangle ABC$, $\angle BCA + x + y = 180°$. (2)
Since $180° = 180°$ in (1) and (2), $\angle BCA + w = \angle BCA + x + y$. $\angle BCA$ is common to both sides. It follows that $x + y = w$.

Therefore, an exterior angle of a triangle is equal to the sum of the opposite interior angles.

Definition:

A *polygon* is a closed plane figure having three or more sides. Triangles, rectangles, and octagons are all examples of polygons. A *regular* polygon is a polygon in which all sides are the same length and all interior angles are the same measure.



From one vertex in each diagram draw line segments to each unconnected vertex. The number of degrees in each figure equals 180° times the number of triangles created.

a) Determine the sum of the interior angles in a quadrilateral.

In each of the four sided figures, two triangles were created. The sum of the interior angles in a quadrilateral is $180° \times 2 = 360°$.

b) Determine the sum of the interior angles in a pentagon.

In the five sided figure, three triangles were created. The sum of the interior angles in a pentagon is $180° \times 3 = 540°$.

c) Determine the sum of the interior angles in a hexagon.

In the six sided figure, four triangles were created. The sum of the interior angles in a hexagon is $180° \times 4 = 720°$.

d) Determine the sum of the interior angles in a heptagon.

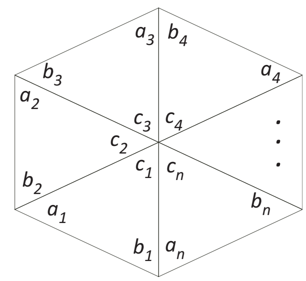
In the seven sided figure, five triangles were created. The sum of the interior angles in a heptagon is $180° \times 5 = 900°$.

In an n-sided figure, $(n - 2)$ triangles were created. The sum of the interior angles in a n-gon is $180° \times (n - 2)$.

This is not a proof but the pattern seems to suggest the result. We will prove this result next.

Prove:

The sum of the interior angles in an n-sided polygon is $180° \times (n-2)$.



Proof:

Place a point in the interior of the n-gon. Connect the point to each vertex. This construction creates n triangles. The total angle sum of all n triangles is $180° \times n$. Label the angles of the triangles as shown.

Expressing the information as an equation:

$$(a_1 + b_1 + c_1) + (a_2 + b_2 + c_2) + (a_3 + b_3 + c_3) + \cdots + (a_n + b_n + c_n) = 180n$$

Rearranging the equation:

$$(a_1 + b_1 + a_2 + b_2 + a_3 + b_3 + \cdots + a_n + b_n) + (c_1 + c_2 + c_3 + \cdots + c_n) = 180n$$

But $(c_1 + c_2 + c_3 + \cdots + c_n)$ forms a complete rotation and therefore equals 360°. Substituting into the equation we obtain:

$$\begin{aligned}(a_1 + b_1 + a_2 + b_2 + a_3 + b_3 + \cdots + a_n + b_n) + (360) &= 180n \\ (a_1 + b_1 + a_2 + b_2 + a_3 + b_3 + \cdots + a_n + b_n) &= 180n - 360, \text{ after rearranging.} \\ (a_1 + b_1 + a_2 + b_2 + a_3 + b_3 + \cdots + a_n + b_n) &= 180(n-2), \text{ after factoring.}\end{aligned}$$

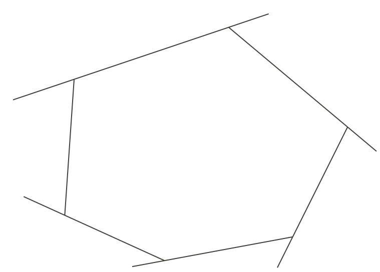
But $(a_1 + b_1 + a_2 + b_2 + a_3 + b_3 + \cdots + a_n + b_n)$ is the sum of the interior angles in the n-gon.

Therefore, the sum of the interior angles in an n-gon is $180° \times (n-2)$.

Since each angle in a regular polygon is equal, then it follows that each angle in a regular n-sided polygon is $\dfrac{180°(n-2)}{n}$.

Prove:

The sum of the exterior angles of an n-sided polygon is 360°.



Proof:

Let the interior angles in the n-gon be $a_1, a_2, a_3, \cdots a_n$ and the adjacent exterior angles be $b_1, b_2, b_3, \cdots b_n$. There are n interior angles and n exterior angles. Each pair of interior and adjacent exterior angles forms a straight line measuring 180°.

It then follows that:

$$(a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdots + (a_n + b_n) = 180n$$

Rearranging:

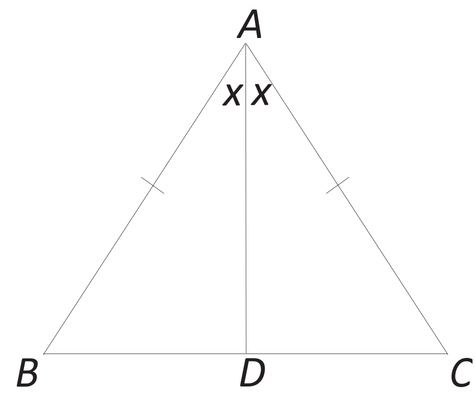
$$\begin{aligned}(a_1 + a_2 + a_3 + \cdots + a_n) + (b_1 + b_2 + b_3 + \cdots + b_n) &= 180n \\ 180(n-2) + (b_1 + b_2 + b_3 + \cdots + b_n) &= 180n \\ 180n - 360 + (b_1 + b_2 + b_3 + \cdots + b_n) &= 180n \\ (b_1 + b_2 + b_3 + \cdots + b_n) &= 360\end{aligned}$$

But $(b_1 + b_2 + b_3 + \cdots + b_n)$ is the sum of the exterior angles.

Therefore the sum of the exterior angles of a polygon is 360°.

Prove:

If a triangle is isosceles, then the angles opposite the equal sides are equal.



Proof:

Construct isosceles $\triangle ABC$ so that $AB = AC$. We are required to prove $\angle ABC = \angle ACB$.

Construct AD, the angle bisector of $\angle BAC$.
It follows that $\angle BAD = \angle DAC$.

In $\triangle BAD$ and $\triangle CAD$, $AB = AC$, $\angle BAD = \angle CAD$ and AD is common. Therefore, $\triangle BAD \cong \triangle CAD$. It follows that $\angle ABC = \angle ACB$, as required.

From the triangle congruency, we get two additional results.

First, $BD = DC$. This means that the angle bisector between the two equal sides of an isosceles triangle bisects the base.

Second, $\angle BDA = \angle CDA$. But $\angle BDA$ and $\angle CDA$ form a straight angle.
So $180^\circ = \angle BDA + \angle CDA = 2\angle BDA$. It then follows that $\angle BDA = 90^\circ$.

Therefore, $\angle BDA = \angle CDA = 90^\circ$ and $BD = DC$.

This means that the angle bisector constructed between the two equal sides of an isosceles triangle right bisects the base. In fact, this bisector is also an altitude. This result is **not** true for the other angle bisectors in an isosceles triangle.