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Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

November 15 & 16, 2016

Areas of Triangles

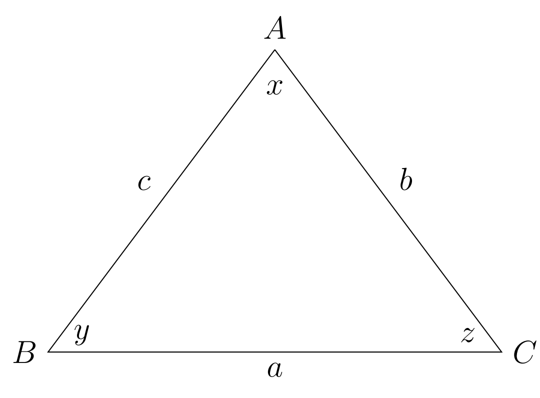
In today's lesson, we are going to learn how to find the area of a triangle in a variety of different ways. Before we get into that, let's review what we know about triangles so far.

Triangles: A Review

A **triangle**, denoted as $\triangle ABC$, has the following properties:

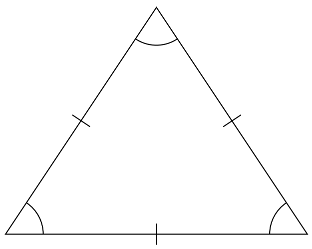
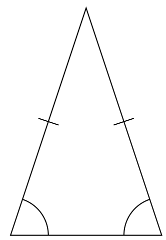
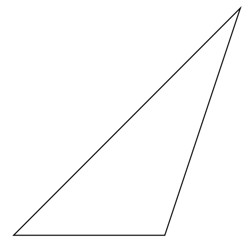
- $\triangle ABC$ has 3 vertices and 3 sides, and 3 angles
- The sum of the interior angles of $\triangle ABC$ is 180°.

We can label a triangle, $\triangle ABC$, as follows:



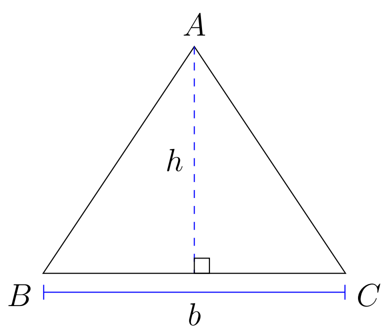
$$\text{Sum of interior angles} = 180^\circ = \angle BAC + \angle ABC + \angle BCA = x + y + z$$

3 Types of Triangles

		
All side lengths are equal All angles are equal	Two side lengths are equal Two angles are equal	All side lengths are different All angles are different

Find the Area of a Triangle

Method 1: Basic Formula

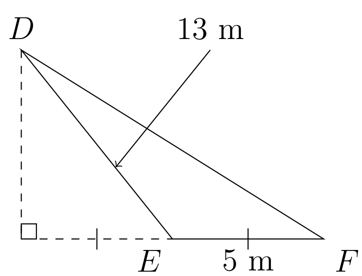


Basic Formula for Area of a Triangle

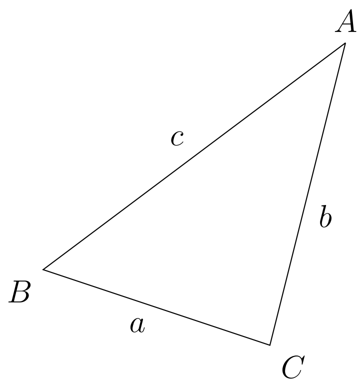
Let b be the base of the triangle and h be the height of the triangle. The area of the triangle, $A_{\triangle ABC}$, is...

$$A_{\triangle ABC} = \frac{b \times h}{2} = \frac{1}{2}(b \times h)$$

Example Find the area of $\triangle DEF$.



Method 2: Heron's Formula

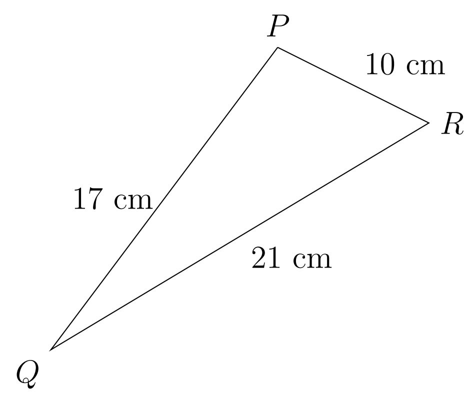


Heron's Formula

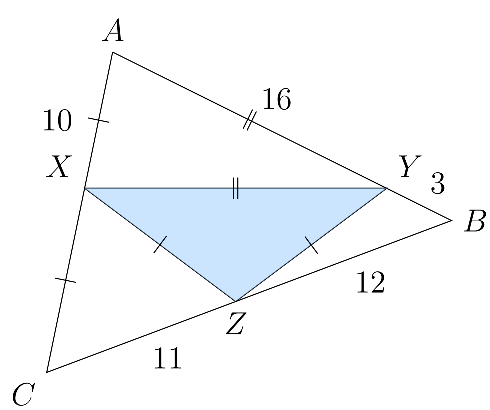
Let a, b, and c be the side lengths of $\triangle ABC$. Then,

where $s = \dfrac{1}{2}(a+b+c)$ is the *semi-perimeter* of $\triangle ABC$.

Example 1 Find the area of $\triangle PQR$:

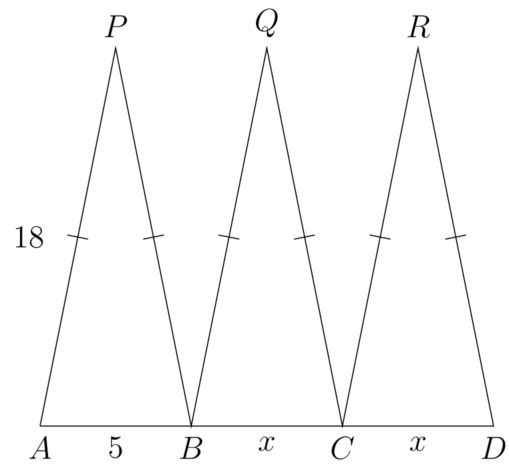


Example 2 Suppose you have $\triangle ABC$. What is the area of $\triangle XYZ$? (Image is not to scale.)



Exercise Challenge

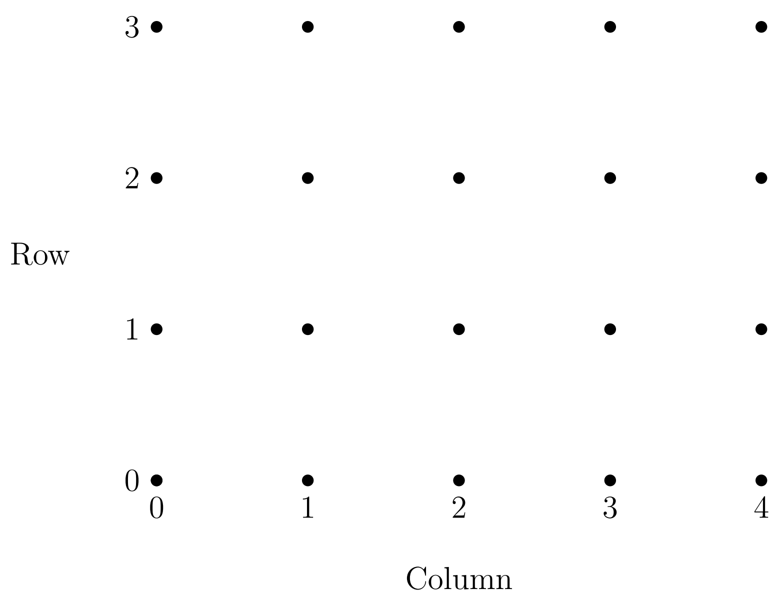
Three identical isosceles triangles are arranged as shown, with points A, B, C, and D lying in the same straight line. If $AR = AD$, what is the area of $\triangle ADR$?



Method 3: Complete the Rectangle

Suppose we have the coordinates of $\triangle ABC$. We do not know what the side lengths are but we are given the coordinates, kind of like the game *Battleship*! Draw the triangle on the graph below.

A(2,3) $\quad$ B(0,0) $\quad$ C(4,1)



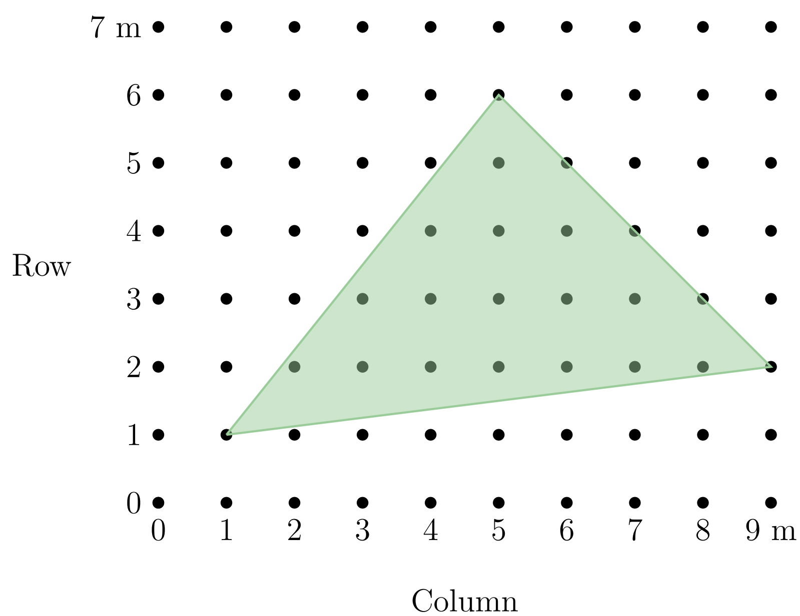
Let's find the area of $\triangle ABC$. Since we do not know the side lengths, we will enclose $\triangle ABC$ inside a rectangle. Notice the rectangle is made up of four triangles: $\triangle ABC$, 1, 2, and 3. How can we calculate the area of $\triangle ABC$?

Complete the Rectangle

If we have a triangle, $\triangle ABC$, and we enclose it in a rectangle, then

where ①, ②, and ③ are triangles that form a rectangle with $\triangle ABC$.

Example Mr. Bean has a 7 m $\times$ 9 m backyard and decides to construct a triangular-shaped patio. He plots his backyard plan on a graph as follows:



What is the area of his patio? How much space does he have left in his backyard?

Method 4: Shoelace Theorem

Also known as "Shoelace Formula," or "Gauss' Area Formula"

Shoelace Theorem (for a Triangle)

Suppose a triangle has the following coordinates: (a_1, b_1), (a_2, b_2), (a_3, b_3) where a_1, a_2, a_3, b_1, b_2, and b_3 can be any positive number. Then,

$$\mathrm{A}_3 = \frac{1}{2}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} =$$

where $|a|$ is called the *absolute value* of a. (i.e. $|-3| = 3$, $|4| = 4$)

Example Given the coordinates below, what is the area of $\triangle ABC$?

$$A(3{,}4) \qquad B(7{,}9) \qquad C(11{,}4)$$

We can actually generalize the Shoelace Theorem and use it to find the area of a shape with any number of sides!

Shoelace Theorem (for a shape with n sides)

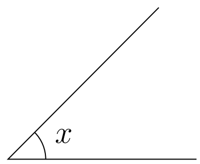
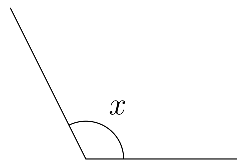
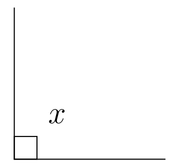
Suppose a n sides, so there are n sets of coordinates: $(a_1, b_1), (a_2, b_2), ..., (a_n, b_n)$. Then,

$$\mathrm{A}_n = \frac{1}{2} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \\ a_1 & b_1 \end{vmatrix} =$$

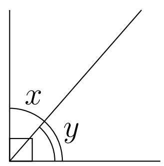
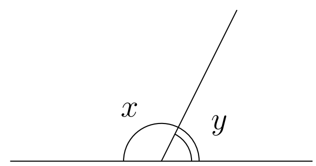
Example Calculate the area of a hexagon with the following coordinates:

(3,9), (9,9), (11,5), (9,1), (3,1), (1,5)

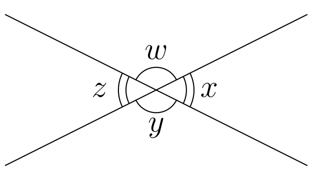
Angles, Angles & More Angles

Acute	Obtuse	Right
		
$x < 90°$	$x > 90°$	$x = 90°$

Complementary vs. Supplementary Angles

Complementary	Supplementary
	
$90° = x + y$	$180° = x + y$

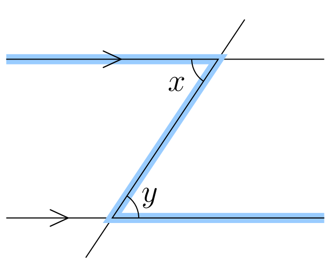
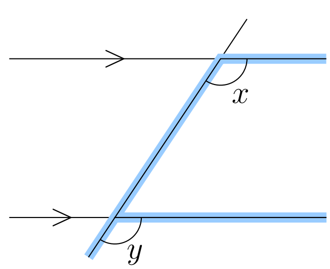
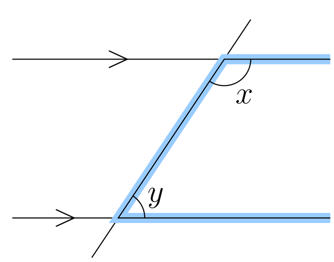
Opposite Angles are the angles opposite of each other when two lines intersect.



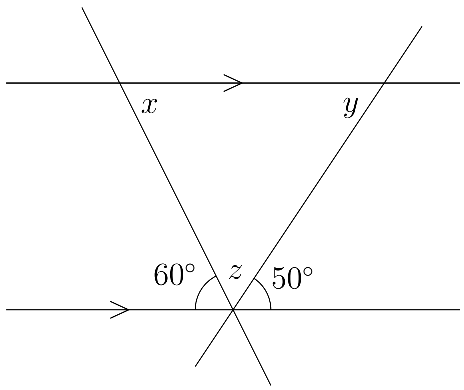
For the example on the left, (w, y) and (x, z) are our pairs of opposite angles. This means that...

$$w = y \qquad x = z$$

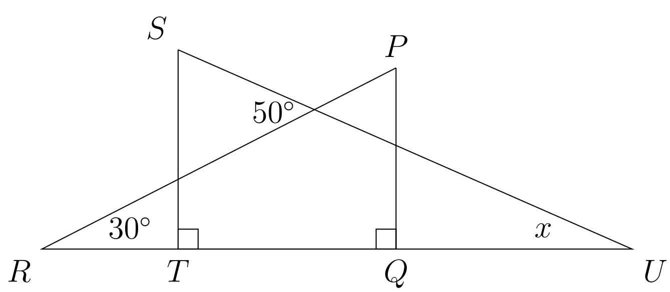
Corresponding Angles occur when there is a line intersecting a pair of parallel lines.

"Z" pattern	"F" pattern	"C" pattern
		
$x = y$	$x = y$	$180° = x + y$

Example 1 Find all the missing angles, x, y, and z, of the following image:
(Note: Image is not to scale. Do not use a protractor.)



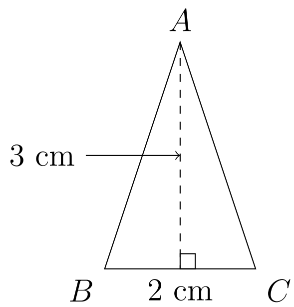
Example 2 In the diagram, $\triangle PQR$ and $\triangle STU$ overlap so that RTQU forms a straight line segment. What is the value of x?



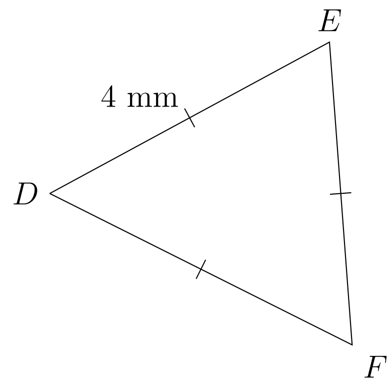
Problem Set

1. Find the area of the following triangles:

   (a)



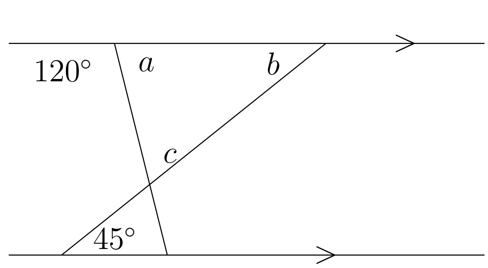
   (b)



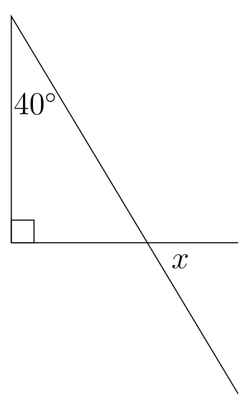
   (c) $\triangle DEF$ where $D(2,8), E(8,0), F(4,4)$

2. Find all the missing angles of the following diagrams and explain your reasoning:

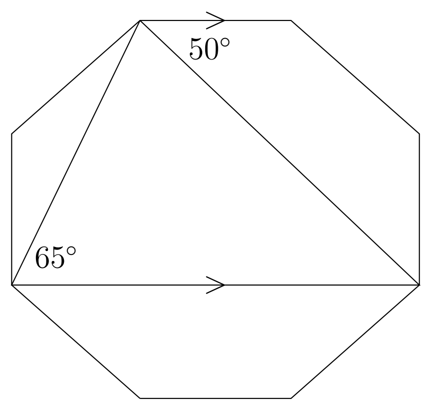
   (a)



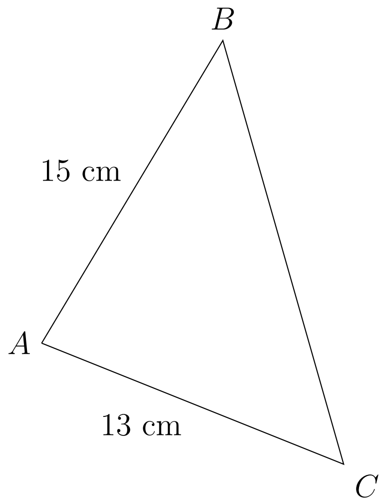
   (b)



   (c)



3. Given the perimeter, $P_{\triangle ABC} = 42$ cm, what is the area of $\triangle ABC$?



4. Given $DE = 12$, $EF = 16$, and $FD = 20$, what is the area of $\triangle DEF$?

5. The perimeter of $\triangle STU$ is 32 cm. If $\angle STU = \angle SUV$ and $TU = 12$ cm, what is the area of $\triangle STU$?

6. Find the area of a rectangle with the following coordinates:

$$(2,3),\ (6,3),\ (6,8),\ (2,8)$$

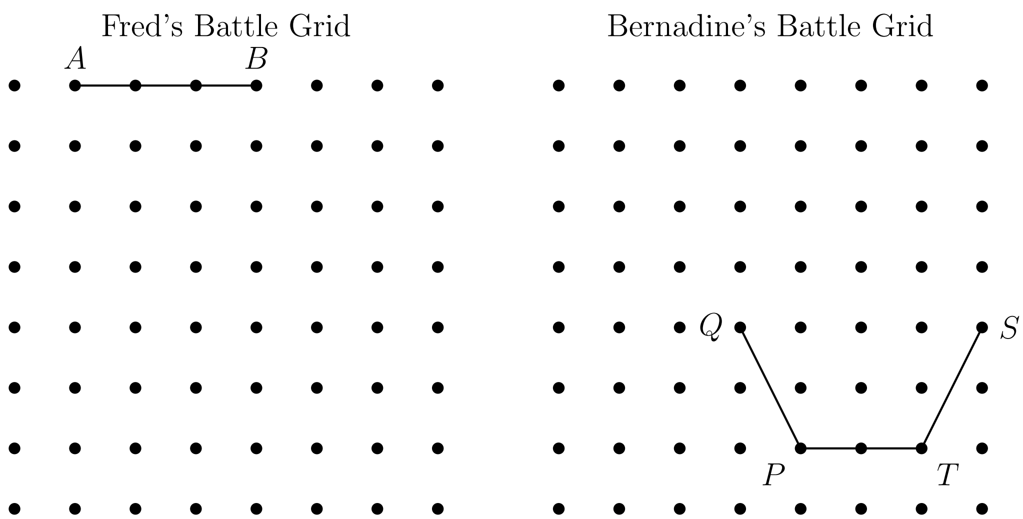
7. Peter Pan and the Lost Boys go looking for treasure in Neverland. They stole the map from Captain Hook but they can read is that the hidden treasure is located somewhere between Mermaid Lagoon (24,10), Pirates Cove (18,7), and Crocodile Creek (20,14). How much area, in kilometres, does Peter Pan and the Lost Boys have to search?

8. * Bernadine and Fred are playing *Battleship: Geometry Edition* where ships sink with one hit. Fred has a triangle-shaped ship with the following coordinates:

$$A(1,7),\ B(4,7),\ C(1,x)$$

and Bernadine has a pentagon-shaped ship with the following coordinates:

$$P(4,1),\ Q(3,3),\ R(5,y),\ S(7,3),\ T(6,1)$$



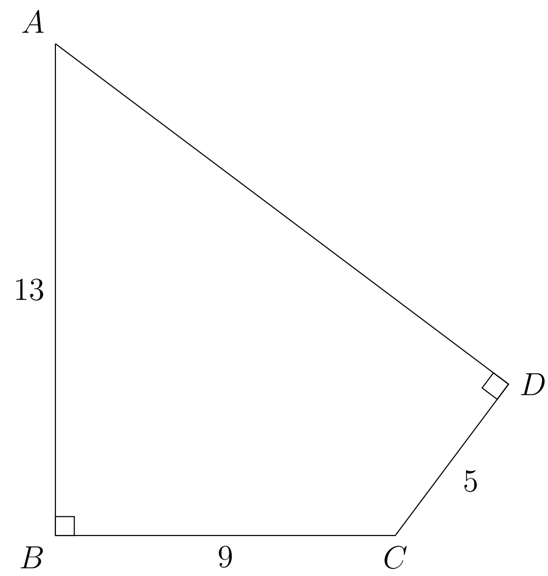
(a) Fred's ship is in the shape of a right-angled triangle. If the area of Fred's ship is 6 units^2, what is the value of x? Draw the rest of Fred's ship on his grid.

(b) Bernadine's ship is in the shape of a pentagon. If the area of Bernadine's ship is 8 units^2, what is the value of y? Draw the rest of Bernadine's ship on her grid.

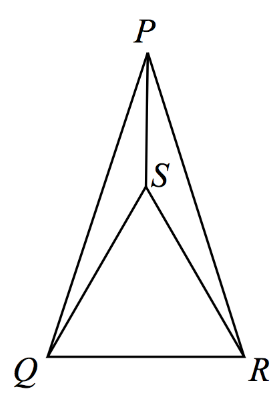
(c) If Fred attacked Bernadine at (4,3), does Fred sink Bernadine's ship?

(d) If Bernadine attacked Fred at (2,6), does Bernadine sink Fred's ship?

9. * In the diagram, what is the area of quadrilateral $ABCD$?



10. ** In the diagram, $\triangle PQR$ is isosceles with $PQ = PR = 39$ and $\triangle SQR$ is equilateral with side length 30. What is the area of $\triangle PQS$? (Round your answer to the nearest whole number.)



11. *** A star is made by overlapping two identical equilateral triangles, as shown below. The entire star has an area of 36. What is the area of the shaded region?