UNIVERSITY OF
WATERLOO

Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

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Areas of Triangles

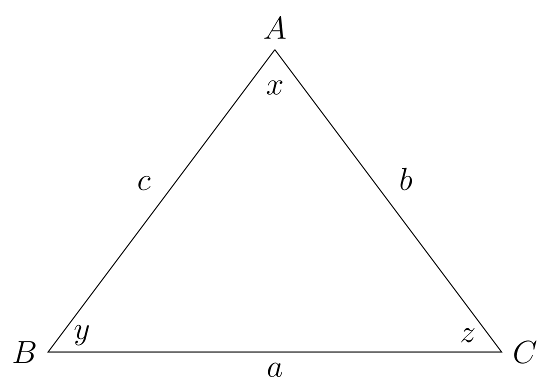
In today's lesson, we are going to learn how to find the area of a triangle in a variety of different ways. Before we get into that, let's review what we know about triangles so far.

Triangles: A Review

A **triangle**, denoted as $\triangle ABC$, has the following properties:

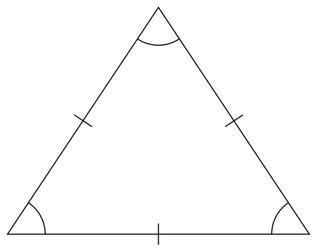
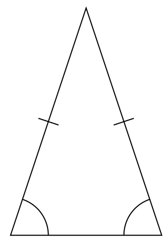
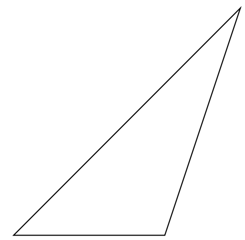
- $\triangle ABC$ has 3 vertices and 3 sides, and 3 angles
- The sum of the interior angles of $\triangle ABC$ is $180°$.

We can label a triangle, $\triangle ABC$, as follows:



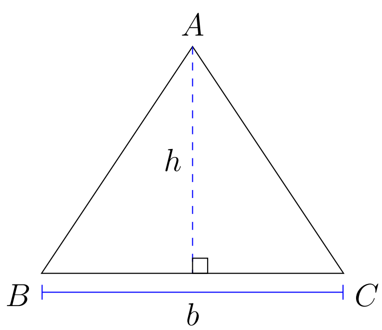
$$\text{Sum of interior angles} = 180° = \angle BAC + \angle ABC + \angle BCA = x + y + z$$

3 Types of Triangles

Equilateral	Isosceles	Scalene
		
All side lengths are equal All angles are equal	Two side lengths are equal Two angles are equal	All side lengths are different All angles are different

Find the Area of a Triangle

Method 1: Basic Formula

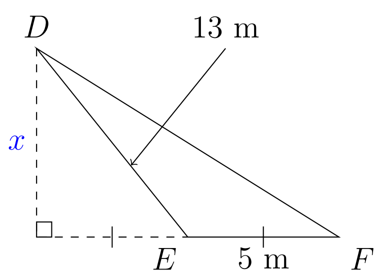


Basic Formula for Area of a Triangle

Let b be the base of the triangle and h be the height of the triangle. The area of the triangle, $A_{\triangle ABC}$, is...

$$A_{\triangle ABC} = \frac{b \times h}{2} = \frac{1}{2}(b \times h)$$

Example Find the area of $\triangle DEF$.

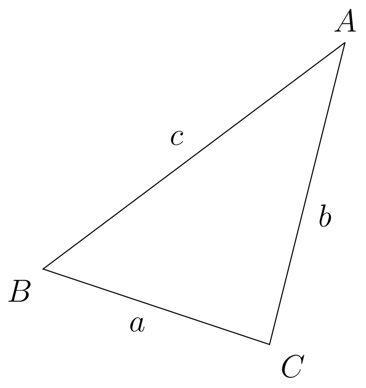


By Pythagorean's Theorem,

$$13^2 = 5^2 + x^2 \Rightarrow x = 12 \text{ m}$$

So, $A_{\triangle DEF} = \dfrac{5 \times 12}{2} = \underline{30\text{m}^2}$

Method 2: Heron's Formula



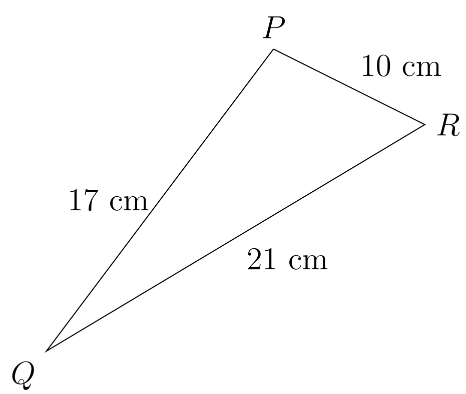
Heron's Formula

Let a, b, and c be the side lengths of $\triangle ABC$. Then,

$$A_{\triangle ABC} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \dfrac{1}{2}(a+b+c)$ is the *semi-perimeter* of $\triangle ABC$.

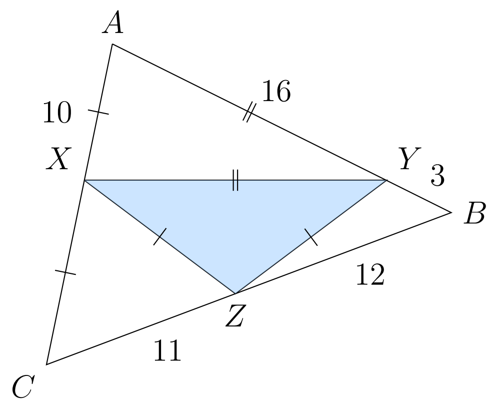
Example 1 Find the area of $\triangle PQR$:



$s = \dfrac{1}{2}(10+21+17) = 24$

$$\begin{aligned} \mathrm{A}_{\triangle PQR} &= \sqrt{24(24-10)(24-21)(24-17)} \\ &= \sqrt{24(14)(3)(7)} \\ &= \sqrt{7056} \\ \mathrm{A}_{\triangle PQR} &= \underline{84 \text{ cm}^2} \end{aligned}$$

Example 2 Suppose you have $\triangle ABC$. What is the area of $\triangle XYZ$? (Image is not to scale.)



Notice that $\triangle AXY$ is isosceles $\Rightarrow XY = 16$. Then, we see that $AX = XC = ZX = YZ = 10$. Now we can use Heron's formula as follows:

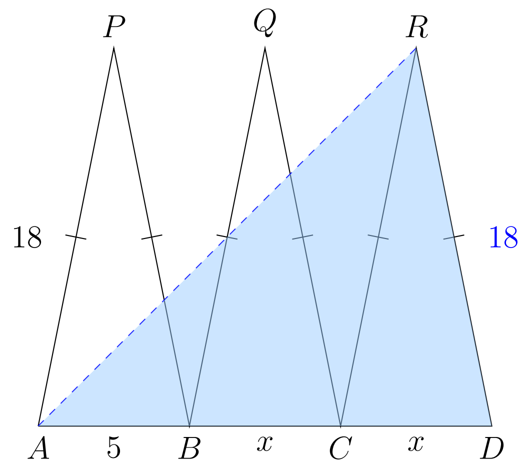
$$s = \frac{1}{2}(XY + YZ + ZX) = \frac{1}{2}(16+10+10) = 18$$

$$\begin{aligned} \mathrm{A}_{\triangle XYZ} &= \sqrt{18(18-16)(18-10)(18-10)} \\ &= \sqrt{18(2)(8)(8)} \\ &= \sqrt{2304} \\ &= 48 \text{ units}^2 \end{aligned}$$

The area of $\triangle XYZ$ is $\underline{48 \text{ units}^2}$.

Exercise Challenge

Three identical isosceles triangles are arranged as shown, with points A, B, C, and D lying in the same straight line. If $AR = AD$, what is the area of $\triangle ADR$?



Since $\triangle APB$, $\triangle BQC$, and $\triangle CRD$ are all identical isosceles triangles, we have that $5 = AB = BC = CD$ and $PA = RD$. Then, we can determine that

$$AD = AB + BC + CD = 5 + 5 + 5 = 15.$$

We also know that $AR = AD$, so $AR = 15$. Now we can use Heron's formula to calculate the area of $\triangle ADR$:

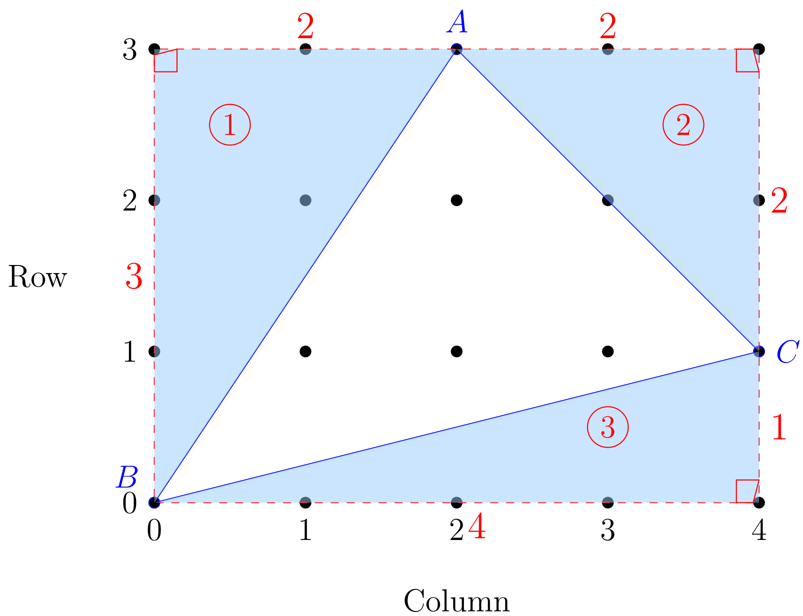
$$\begin{aligned} s &= \frac{1}{2}(15 + 15 + 18) = 24 \\ \mathrm{A}_{\triangle ADR} &= \sqrt{24(24-15)(24-15)(24-18)} \\ &= \sqrt{24(9)(9)(6)} \\ &= \sqrt{11664} \\ \mathrm{A}_{\triangle ADR} &= 108 \text{ units}^2 \end{aligned}$$

The area of $\triangle ADR$ is $\underline{108 \text{ units}^2}$.

Method 3: Complete the Rectangle

Suppose we have the coordinates of $\triangle ABC$. We do not know what the side lengths are but we are given the coordinates, kind of like the game *Battleship*! Draw the triangle on the graph below.

A(2,3) $\qquad B$(0,0) $\qquad C$(4,1)



Let's find the area of $\triangle ABC$. Since we do not know the side lengths, we will enclose $\triangle ABC$ inside a rectangle. Notice the rectangle is made up of four triangles: $\triangle ABC$, 1, 2, and 3. How can we calculate the area of $\triangle ABC$?

Subtract the areas of 1, 2, and 3 from the area of the rectangle!

$$
\begin{aligned}
\mathrm{A}_{rectangle} &= \text{length} \times \text{width} = 4 \times 3 = 12 \\
\mathrm{A}_{1} &= \frac{b \times h}{2} = \frac{2 \times 3}{2} = 3 \\
\mathrm{A}_{2} &= \frac{2 \times 2}{2} = 2 \\
\mathrm{A}_{3} &= \frac{4 \times 1}{2} = 2 \\
\mathrm{A}_{\triangle ABC} &= \mathrm{A}_{rectangle} - \mathrm{A}_{1} - \mathrm{A}_{2} - \mathrm{A}_{3} \\
\mathrm{A}_{\triangle ABC} &= 12 - 3 - 2 - 2 \\
\mathrm{A}_{\triangle ABC} &= \underline{5 \text{ units}^2}
\end{aligned}
$$

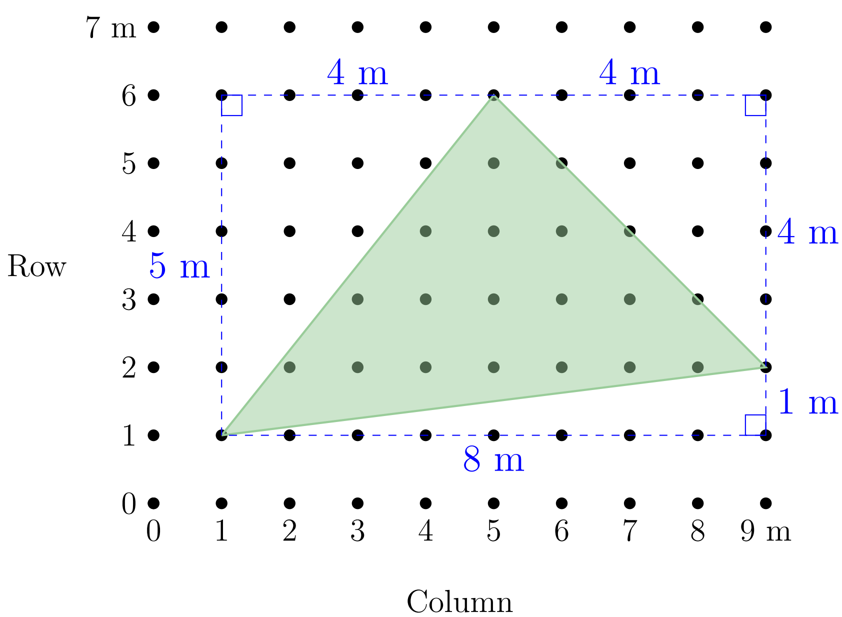
Complete the Rectangle

If we have a triangle, $\triangle ABC$, and we enclose it in a rectangle, then

$$\mathrm{A}_{\triangle ABC} = \mathrm{A}_{rectangle} - \mathrm{A}_{①} - \mathrm{A}_{②} - \mathrm{A}_{③}$$

where ①, ②, and ③ are triangles that form a rectangle with $\triangle ABC$.

Example Mr. Bean has a 7 m × 9 m backyard and decides to construct a triangular-shaped patio. He plots his backyard plan on a graph as follows:



What is the area of his patio? How much space does he have left in his backyard?

$$\begin{aligned}
\mathrm{A}_{patio} &= \mathrm{A}_{rectangle} - \mathrm{A}_{①} - \mathrm{A}_{②} - \mathrm{A}_{③} \\
&= (5 \times 8) - \frac{5 \times 4}{2} - \frac{4 \times 4}{2} - \frac{8 \times 1}{2} \\
&= 40 - 10 - 8 - 4 \\
&= 18 \text{ m}^2
\end{aligned}$$

The area of Mr. Bean's patio is $\underline{18 \text{ m}^2}$.

After building his patio, he will have $(7 \times 9) - 18 = 63 - 18 = \underline{45 \text{ m}^2}$ of his backyard left.

Method 4: Shoelace Theorem

Also known as "Shoelace Formula," or "Gauss' Area Formula"

Shoelace Theorem (for a Triangle)

Suppose a triangle has the following coordinates: (a_1, b_1), (a_2, b_2), (a_3, b_3) where a_1, a_2, a_3, b_1, b_2, and b_3 can be any positive number. Then,

$$\mathrm{A}_3 = \frac{1}{2}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} = \frac{1}{2}\left|(a_2b_1 + a_3b_2 + a_1b_3) - (a_1b_2 + a_2b_3 + a_3b_1)\right|$$

where $|a|$ is called the *absolute value* of a. (i.e. $|-3| = 3$, $|4| = 4$)

Example Given the coordinates below, what is the area of $\triangle ABC$?

$$A(3{,}4) \qquad B(7{,}9) \qquad C(11{,}4)$$

$$\begin{aligned} \mathrm{A}_3 &= \frac{1}{2}\begin{vmatrix} 3 & 4 \\ 7 & 9 \\ 11 & 4 \\ 3 & 4 \end{vmatrix} \\ &= \frac{1}{2}\left|(7\times 4 + 11\times 9 + 3\times 4) - (3\times 9 + 7\times 4 + 11\times 4)\right| \\ &= \frac{1}{2}\left|(28 + 99 + 12) - (27 + 28 + 44)\right| \\ &= \frac{1}{2}\left|139 - 99\right| \\ &= \frac{1}{2}\left|40\right| \\ &= \frac{1}{2}\times 40 \\ \mathrm{A}_3 &= 20\ \mathrm{m}^2 \end{aligned}$$

The area of $\triangle ABC$ is $\underline{20\ \mathrm{m}^2}$.

We can actually generalize the Shoelace Theorem and use it to find the area of a shape with any number of sides!

Shoelace Theorem (for a shape with n sides)

Suppose a n sides, so there are n sets of coordinates: $(a_1, b_1), (a_2, b_2), ..., (a_n, b_n)$. Then,

$$\mathrm{A}_n = \frac{1}{2}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \\ a_1 & b_1 \end{vmatrix} = \frac{1}{2}\left|(a_2b_1 + a_3b_2 + \ldots + a_1b_n) - (a_1b_2 + a_2b_3 + \ldots + a_nb_1)\right|$$

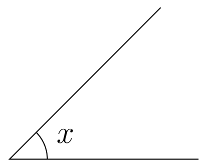
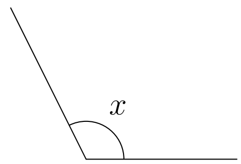
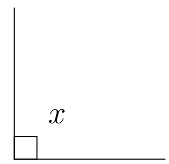
Example Calculate the area of a hexagon with the following coordinates:

(3,9), (9,9), (11,5), (9,1), (3,1), (1,5)

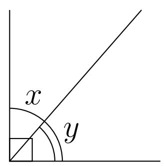
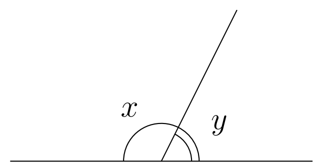
$$\begin{aligned}
\mathrm{A}_6 &= \frac{1}{2}\begin{vmatrix} 3 & 9 \\ 9 & 9 \\ 11 & 5 \\ 9 & 1 \\ 3 & 1 \\ 1 & 5 \\ 3 & 9 \end{vmatrix} \\
&= \frac{1}{2}|(9\times 9 + 11\times 9 + 9\times 5 + 3\times 1 + 1\times 1 + 3\times 5) \\
&\quad -(3\times 9 + 9\times 5 + 11\times 1 + 9\times 1 + 3\times 5 + 1\times 9)| \\
&= \frac{1}{2}\left|(81 + 99 + 45 + 3 + 1 + 15) - (27 + 45 + 11 + 9 + 15 + 9)\right| \\
&= \frac{1}{2}\left|244 - 116\right| \\
&= \frac{1}{2}\left|128\right| \\
&= \frac{1}{2}\times 128 \\
\mathrm{A}_6 &= 64 \text{ units}^2
\end{aligned}$$

The area of the hexagon is $\underline{64 \text{ units}^2}$.

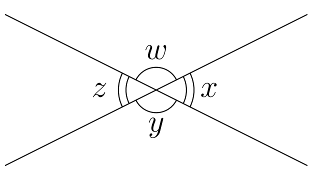
Angles, Angles & More Angles

Acute	Obtuse	Right
		
$x < 90°$	$x > 90°$	$x = 90°$

Complementary vs. Supplementary Angles

Complementary	Supplementary
	
$90° = x + y$	$180° = x + y$

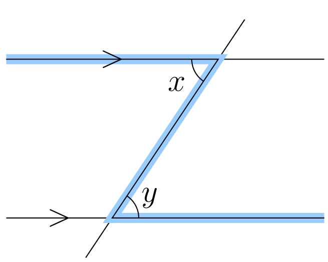
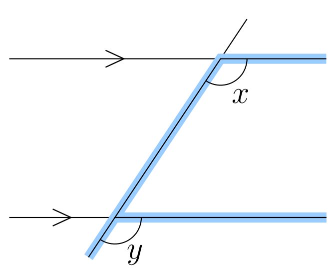
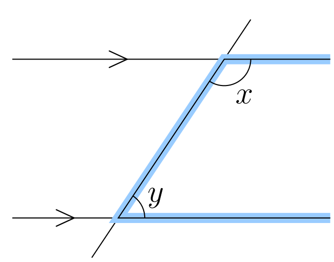
Opposite Angles are the angles opposite of each other when two lines intersect.



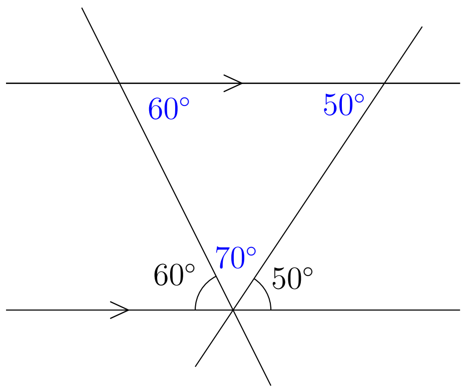
For the example on the left, $(w,\, y)$ and $(x,\, z)$ are our pairs of opposite angles. This means that...

$$w = y \qquad\qquad x = z$$

Corresponding Angles occur when there is a line intersecting a pair of parallel lines.

"Z" pattern	"F" pattern	"C" pattern
		
$x = y$	$x = y$	$180° = x + y$

Example 1 Find all the missing angles, x, y, and z, of the following image:
(Note: Image is not to scale. Do not use a protractor.)



Solutions may vary.
Using supplementary angles, we know

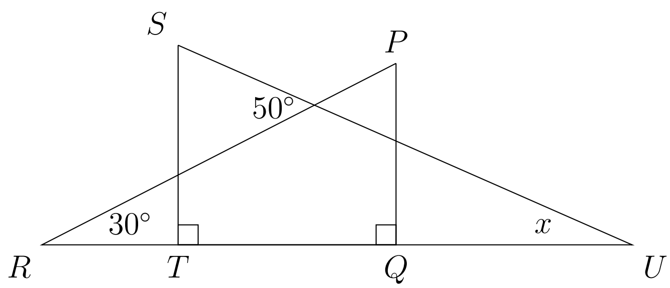
$$180° = 60° + z + 50° \Rightarrow z = 70°$$

Then, by the "Z" pattern, we see that $y = 50°$
Finally, by the sum of interior angles, we also know that...

$$180° = x + 50° + 70° \Rightarrow x = 60°$$

So, $x = 60°$ $\quad y = 50°$ $\quad z = 70°$

Example 2 In the diagram, $\triangle PQR$ and $\triangle STU$ overlap so that RTQU forms a straight line segment. What is the value of x?



The supplementary angle where PR intersects SU is $180° - 50° = 130°$. And since the sum of interior angles is 180°, we have that

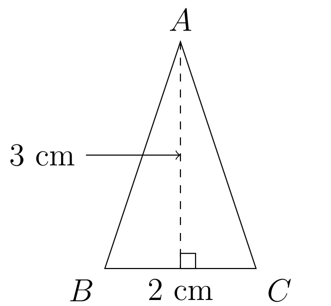
$$180° = 30° + 130° + x \Rightarrow x = 180° - 30° - 130° = 20°$$

Therefore, $\underline{x = 20°}$.

Problem Set Solutions

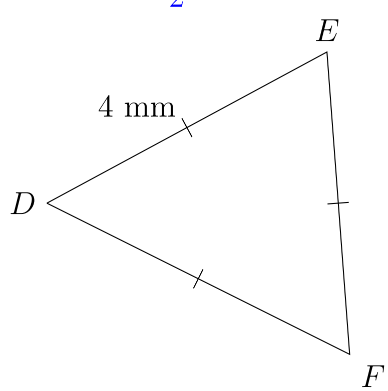
1. Find the area of the following triangles:

   (a)

   $$A_{\triangle ABC} = \frac{2 \times 3}{2} = 3 \text{ cm}^2$$

   (b)

   Use Heron's Formula:

   $$s = \frac{1}{2}(4 + 4 + 4) = 6$$

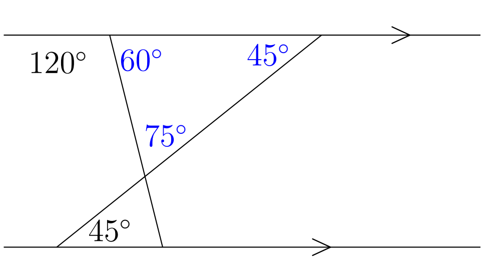
   $$\mathrm{A}_{\triangle DEF} = \sqrt{6(2)(2)(2)} = \sqrt{48} \approx 6.928 \text{ mm}^2$$

   (c) $\triangle DEF$ where $D(2, 8), E(8, 0), F(4, 4)$

   $$\begin{aligned}
\mathrm{A}_{\triangle DEF} &= \frac{1}{2}\begin{vmatrix} 2 & 8 \\ 8 & 0 \\ 4 & 4 \\ 2 & 8 \end{vmatrix} = \frac{1}{2}\left|(8 \times 8 + 4 \times 0 + 2 \times 4) - (2 \times 0 + 8 \times 4 + 4 \times 8)\right| \\
&= \frac{1}{2}\left|72 - 64\right| \\
&= \frac{1}{2}\left|8\right| \\
\mathrm{A}_{\triangle DEF} &= 4 \text{ units}^2
\end{aligned}$$

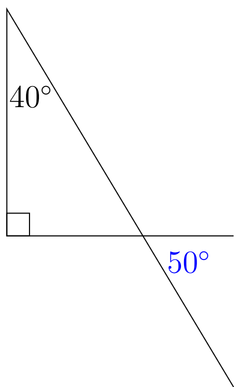
2. Find all the missing angles of the following diagrams and explain your reasoning:
   Reasoning for solutions may vary.

   (a)



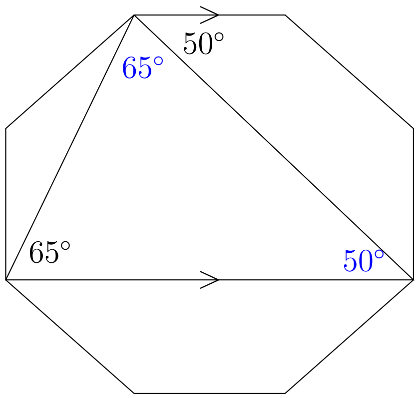
   Since a is a supplementary angle of $120°$, $a = 60°$. By the "Z" pattern, $b = 45°$. Using the sum of interior angles, $c = 180° - 60° - 45° - 75°$.

   (b)



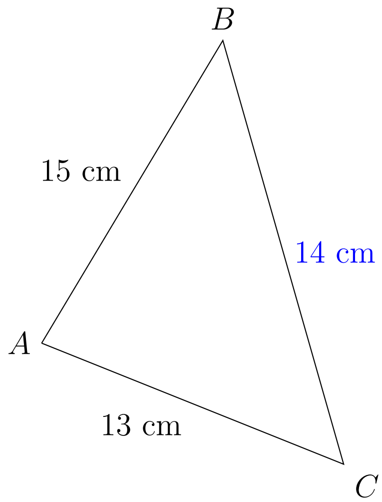
   By the sum of interior angles, the missing angle of the triangle is $180° - 90° - 40° = 50°$. Using opposite angles, $x = 50°$.

   (c)



   By the "C" pattern, $p = 180° - 65° - 50° = 65°$. By the "Z" pattern, $q = 50°$.

3. Given the perimeter, $\mathrm{P}_{\triangle ABC} = 42$ cm, what is the area of $\triangle ABC$?



First, we need to find x using the perimeter.

$$\mathrm{P}_{\triangle ABC} = 42 = x + 15 + 13 \Rightarrow x = 14 \text{ cm}$$

Now, we can use Heron's formula to find the area.

$$s = \frac{1}{2}(14 + 15 + 13) = 21 \text{ cm}$$
$$\mathrm{A}_{\triangle ABC} = \sqrt{21(21-14)(21-15)(21-13)} = \sqrt{21(7)(6)(8)} = \sqrt{7056} = 84 \text{ cm}^2$$

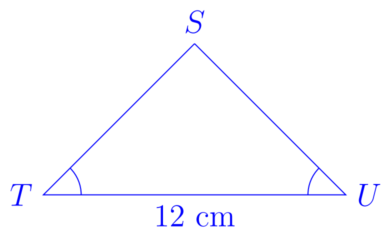
4. Given $DE = 12$, $EF = 16$, and $FD = 20$, what is the area of $\triangle DEF$?

Using Heron's Formula, we get:

$$s = \frac{1}{2}(12 + 16 + 20) = 24$$
$$\mathrm{A}_{\triangle DEF} = \sqrt{24(24-12)(24-16)(24-20)} = \sqrt{24(12)(8)(4)} = \sqrt{9216} = 96 \text{ units}^2$$

5. The perimeter of $\triangle STU$ is 32 cm. If $\angle STU = \angle SUV$ and $TU = 12$ cm, what is the area of $\triangle STU$?

   Drawing $\triangle STU$ gives us:



   Since $\angle STU = \angle SUV$, $\triangle STU$ is an isosceles triangle which means that $ST = US$. Use the perimeter to solve for ST and US.

$$\mathrm{P}_{\triangle STU} = 32 = 12 + ST + US = 12 + 2ST \Rightarrow ST = \frac{32-12}{2} = 10 \text{ cm}$$

   Using Heron's Formula, we can find the area of $\triangle STU$:

$$s = \frac{1}{2}(12 + 10 + 10) = 16$$
$$\mathrm{A}_{\triangle STU} = \sqrt{16(16-12)(16-10)(16-10)} = \sqrt{16(4)(6)(6)} = \sqrt{2304} = 48 \text{ cm}^2$$

6. Find the area of a rectangle with the following coordinates:

   (2,3), (6,3), (6,8), (2,8)

   Using Shoelace Theorem, we get that...

$$\begin{aligned}
\mathrm{A}_4 &= \frac{1}{2}\begin{vmatrix} 2 & 3 \\ 6 & 3 \\ 6 & 8 \\ 2 & 8 \\ 2 & 3 \end{vmatrix} \\
&= \frac{1}{2}\left|(6\times 3 + 6\times 3 + 2\times 8 + 2\times 8) - (2\times 3 + 6\times 8 + 6\times 8\times 2\times 3)\right| \\
&= \frac{1}{2}\left|68 - 108\right| \\
&= \frac{1}{2}\left|-40\right| \\
&= \frac{1}{2}(40) \\
\mathrm{A}_4 &= 20 \text{ units}^2
\end{aligned}$$

7. Peter Pan and the Lost Boys go looking for treasure in Neverland. They stole the map from Captain Hook but they can read is that the hidden treasure is located somewhere between Mermaid Lagoon (24,10), Pirates Cove (18,7), and Crocodile Creek (20,14). How much area, in kilometres, does Peter Pan and the Lost Boys have to search?

   Use the Shoelace Theorem!

$$
\begin{aligned}
A_3 &= \frac{1}{2}\begin{vmatrix} 24 & 10 \\ 18 & 7 \\ 20 & 14 \\ 24 & 10 \end{vmatrix} \\
&= \frac{1}{2}\left|(18\times 10 + 20\times 7 + 24\times 14) - (24\times 7 + 18\times 14 + 20\times 10)\right| \\
&= \frac{1}{2}\left|656 - 620\right| \\
&= \frac{1}{2}\left|36\right| \\
&= \frac{1}{2}(36) \\
A_3 &= 18\ \text{km}^2
\end{aligned}
$$

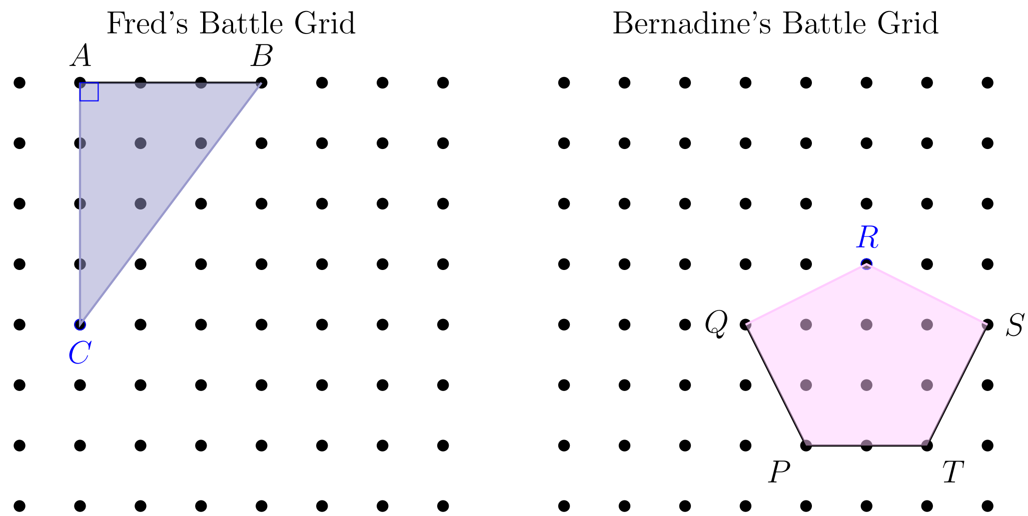
   Peter Pan and the Lost Boys has <u>18 km^2</u> of area to search.

8. * Bernadine and Fred are playing *Battleship: Geometry Edition* where ships sink with one hit. Fred has a triangle-shaped ship with the following coordinates:

$$A(1,7),\ B(4,7),\ C(1,x)$$

and Bernadine has a pentagon-shaped ship with the following coordinates:

$$P(4,1),\ Q(3,3),\ R(5,y),\ S(7,3),\ T(6,1)$$



(a) Fred's ship is in the shape of a right-angled triangle. If the area of Fred's ship is 6 units^2, what is the value of x? Draw the rest of Fred's ship on his grid.

Since we know the area is 6 units^2 and the triangle is a right-angled triangle, we can use the basic formula for area to find the height, $h(= BC)$. We can say AB $(= 3)$ units is the base of our triangle (i.e. $b = AB$).

$$\begin{aligned} \mathrm{A}_{\triangle ABC} = 6 &= \frac{3 \times h}{2} \\ 2\times\ 6 &= \frac{3 \times h}{\cancel{2}} \times \cancel{2} \\ \frac{12}{3} &= \frac{\cancel{3} \times h}{\cancel{3}} \\ 4 \text{ units} &= h \end{aligned}$$

Since the height of Fred's triangle-shaped ship is 4 units, $x = 7 - 4 = 3$. Thus, $C(1, x) = C(1,3)$.

(b) Bernadine's ship is in the shape of a pentagon. If the area of Bernadine's ship is 8 units^2, what is the value of y? Draw the rest of Bernadine's ship on her grid.

Since we know the area of the pentagon-shaped ship is 8 units^2 and we know all coordinates except for R, we can use the Shoelace Theorem and solve for y.

$$\begin{aligned}
\mathrm{A}_{PQRST} &= \frac{1}{2}\begin{vmatrix} 4 & 1 \\ 3 & 3 \\ 5 & y \\ 7 & 3 \\ 6 & 1 \\ 4 & 1 \end{vmatrix} \\
8 &= \frac{1}{2}|(3\times 1+5\times 3+7\times y+6\times 3+4\times 1) \\
&\quad -(4\times 3+3\times y+5\times 3+7\times 1+6\times 1)| \\
8 &= \frac{1}{2}\left|(7y+\cancel{40})-(3y+\cancel{40})\right| \\
8 &= \frac{1}{2}\left|4y\right| \\
2\times\ 8 &= \frac{4y}{\cancel{2}}\times\cancel{2} \\
\frac{16}{4} &= \frac{\cancel{4}y}{\cancel{4}} \\
4 &= y
\end{aligned}$$

Therefore, $y = 4$.

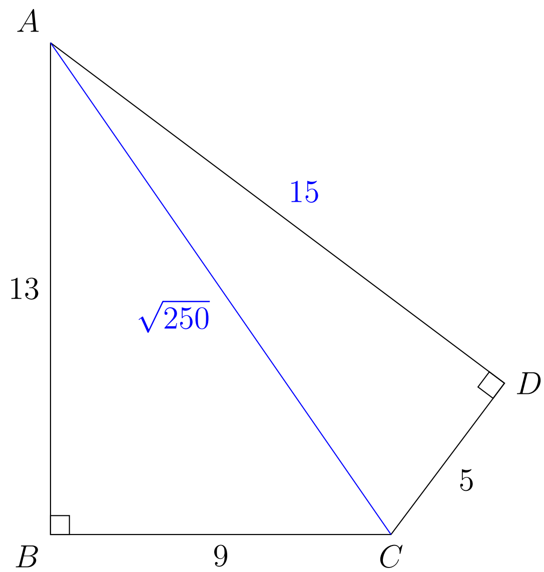
(c) If Fred attacked Bernadine at (4,3), does Fred sink Bernadine's ship?

No, he does not. Unfortunately, Fred has poor aim.

(d) If Bernadine attacked Fred at (2,6), does Bernadine sink Fred's ship?

Yes, Fred's ship sinks.

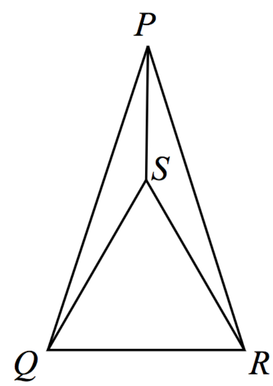
9. * In the diagram, what is the area of quadrilateral $ABCD$?



Draw line segment AC to create two right-angled triangles, $\triangle ABC$ and $\triangle ACD$. By Pythagorean's Theorem, $AC = \sqrt{9^2 + 13^2} = \sqrt{250} \approx 15.81$. Again, by Pythagorean's Theorem, $AD = \sqrt{AC^2 - 5^2} = \sqrt{250 - 25} = \sqrt{225} = 15$. The area of $ABCD$ is the sum of the areas of $\triangle ABC$ and $\triangle ACD$. Both triangles are right-angled, so we can use the basic formula to find the area of both triangles.

$$\mathrm{A}_{ABCD} = \mathrm{A}_{\triangle ABC} + \mathrm{A}_{\triangle ACD} = \frac{9 \times 13}{2} + \frac{5 \times 15}{2} = \frac{117 + 75}{2} = \frac{192}{2} = 96 \text{ units}^2$$

10. ** In the diagram, $\triangle PQR$ is isosceles with $PQ = PR = 39$ and $\triangle SQR$ is equilateral with side length 30. What is the area of $\triangle PQS$? (Round your answer to the nearest whole number.)



To find the area of $\triangle PQS$, we will calculate the area of $PQSR$ by subtracting the area of $\triangle SQR$ from $\triangle PQR$. Use Heron's formula to calculate the area of both triangles.

$$s_{\triangle PQR} = \frac{1}{2}(39 + 39 + 30) = 54$$
$$\text{A}_{\triangle PQR} = \sqrt{54(54-39)(54-39)(54-30)} = \sqrt{54(15)(15)(24)} = \sqrt{291600} = 540$$

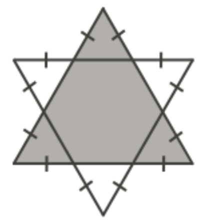
$$s_{\triangle SQR} = \frac{1}{2}(30 + 30 + 30) = 45$$
$$\text{A}_{\triangle SQR} = \sqrt{45(45-30)(45-30)(45-30)} = \sqrt{45(15)(15)(15)} = \sqrt{3375} \approx 389.71$$

Now, $\text{A}_{PQSR} = \text{A}_{\triangle PQR} - \text{A}_{\triangle SQR} = 540 - 389.71 = 150.29$. The area of $\triangle PQS$ is half of the area of $PQSR$ since $PQ = PR$, $SQ = SR$, and $\triangle PQS$ shares side length PS with $\triangle PRS$.

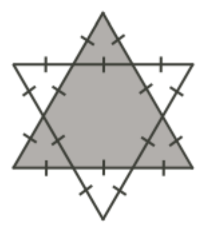
$$\text{A}_{\triangle PQS} = \frac{1}{2} \times \text{A}_{PQSR} = \frac{1}{2} \times 150.29 \approx 75.145 = 75$$

Therefore, the area of $\triangle PQS$ is 75 units^2.

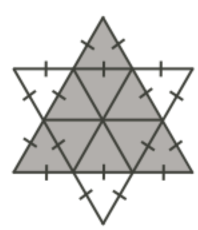
11. *** A star is made by overlapping two identical equilateral triangles, as shown below. The entire star has an area of 36. What is the area of the shaded region?



Since the triangles are equilateral, each angle of both triangles is $60°$. Each smaller triangle has two equal side lengths, marked by the single tick, so all 6 of the smaller triangles are identical isosceles triangles. Since the smaller triangles are isosceles, the missing angles can be calculated as follows: $\frac{1}{2}(180° - 60°) = 60°$. Because each angle of the small triangle is $60°$, all the small triangles are identical equilateral triangles as shown below.



Now, all side lengths of the inner hexagon are equal, so it is a regular hexagon. This mean each angle is equal to $120°$. Since a regular hexagon is symmetrical, drawing the 3 diagonals of the hexagon creates 6 more small triangles as shown below.



As it turns out, the small triangles in the inner hexagon are also identical equilateral triangles. (The diagonals split the angles of the hexagon in half, $120° \div 2 = 60°$. $\Rightarrow$ The third angle in the small triangles also equal $60°$.) The star is made of 12 small identical equilateral triangles so the area of each small triangle is $36 \div 12 = 3$. The shaded region is made of 9 small triangles and so the area of the shaded region is $9 \times 3 = \underline{27}$.