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Grade 7/8 Math Circles

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Combinatorial Counting

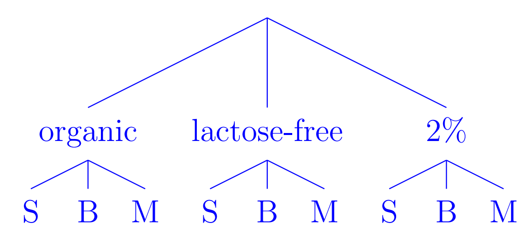
Learning How to Count (In a New Way!)

Right now, you are probably thinking, "Counting? I know how to count! What kind of a lesson is this? What is this madness???" How about this: in today's lesson, we will learn about different ways of counting things, how to count the number of different possibilities and outcomes of actions or events. Counting actually falls under a branch of mathematics called **combinatorics**. By the end of this lesson, you will be the Master Yoda of counting sharing your knowledge and wisdom with your own young padawans.

Warm Up

Make a Smoothie

You're making yourself a smoothie after a long day at school. In the kitchen, you have 3 different types of milk: organic, lactose-free, or 2% and 3 different fruits: strawberries, bananas, or mangoes. How many different smoothies can you make?



There are 9 possible outcomes.

Product Rule (Fundamental Counting Principle)

Product Rule (or the Fundamental Counting Principle)
If there are m ways of doing one action and n ways of doing another action, then there are $m \times n$ ways of doing both actions.

Example It's getting chilly outside and you have to choose which hat and scarf you want to wear. You have a plain red hat and a striped blue hat. And you have a plain blue scarf, striped green scarf, and a spotted red scarf. How many different hat and scarf combinations can you wear?

2 hats × 3 scarves
= 6 different hat and scarf combinations

Sum Rule

Sum Rule
If there are m ways of doing one action and n ways of doing another action, then there are $m + n$ ways of doing one action or the other action.
(Note: Both actions cannot be done together)

Example You are at the movie theatre to watch Doctor Strange with your friends. You have enough money to buy popcorn or nachos. There are three sizes of popcorn: small, medium, and large. And there are three kinds of nachos: cheese, chili cheese, or no cheese. How many movie snack options do you have?

3 popcorn sizes + 3 kinds of nachos = 6 movie snacks in total

Factorials

Try this! Suppose you are arranging three balls in a line: one red, one blue, and one yellow ball. How many different arrangements are there?

There are 6 different arrangements.

Let n be a whole number. The **factorial** of a whole number, denoted as $n!$, is the product of all whole numbers less than or equal to n.

$$n! = (n) \times (n-1) \times (n-2) \times ... \times (2) \times (1)$$

We read this as "n-factorial".

Example 1 Calculate the following:

(a) $3! = 3 \times 2 \times 1 = 6$ (b) $4! = 4 \times 3 \times 2 \times 1 = 24$ (c) $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

It may be helpful to note that...

$$n! = n \times (n-1)!$$

We can see this in true in the previous examples.

Example 2 You and 5 other friends are watching the Toronto Maple Leafs game. In how many different ways can you and your friends sit in your seats?

There are 6 of you in total and 6 seats. In the first seat, there are 6 people who can take seat (6 possible actions). One person sits down. In the second seat, there are now 5 people who can take the seat (5 possible actions). In the third seat, there are 4 people who can take the seat. For the fourth seat, there are 3 people. The fifth seat, there are 2 people. And for the sixth seat, there is only one person left. So, we can write this as a factorial....

$$\underline{\ 6\ } \times \underline{\ 5\ } \times \underline{\ 4\ } \times \underline{\ 3\ } \times \underline{\ 2\ } \times \underline{\ 1\ }$$

There are $6! =$ 720 different ways you and your friends can sit to watch the Leafs game.

A few notes about factorials...

- 0! is a special case. We can say that 0! = 1
- We can only find the factorials of positive integers (or positive whole numbers) (i.e. π!, 2.81!, and (-3)! are **not** possible to find)

Multiplying and Dividing Factorials

Yes, it is possible to multiply and divide factorials! Take a look at the following examples.

Multiplying Factorials

Calculate 3! × 2!

$$3! \times 2! = (3 \times 2 \times 1)(2 \times 1) = (6)(2) = 12$$

Nothing really new here with multiplying.

Dividing Factorials

Dividing factorials is a little more interesting. Calculate the following: $\frac{7!}{4!}$

$$\begin{aligned}
\frac{7!}{4!} &= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} \\
&= \frac{7 \times 6 \times 5 \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} \\
&= 7 \times 6 \times 5 \\
&= 210
\end{aligned}$$

Example Calculate $\frac{12!}{13!}$

$$\frac{12!}{13!} = \frac{\cancel{12} \times \cancel{11} \times \cancel{10} \times \cancel{9} \times \cancel{8} \times \cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{13 \times \cancel{12} \times \cancel{11} \times \cancel{10} \times \cancel{9} \times \cancel{8} \times \cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} = \frac{1}{13}$$

OR

$$\frac{12!}{13!} = \frac{\cancel{12!}}{13 \times \cancel{12!}} = \frac{1}{13}$$

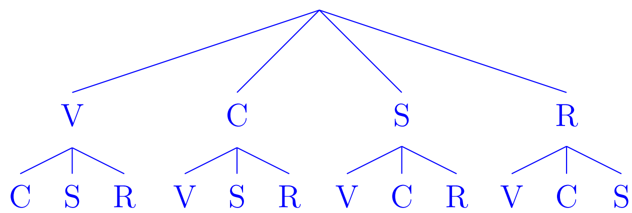
Permutations (Order Does Matter!)

Consider the following problem:

You're making a two-layer cake and have four flavours of cake to choose from: vanilla, chocolate, strawberry, and red velvet.

Question: How many different two-layer cakes can you make?
(Suppose each layer is different and you cannot repeat a flavour.)

Answer this by making a tree diagram.



The first row represents the top layer of the cake and the second row of flavours represents the bottom layer of cake.

Answer: In total, there are 12 different two-layer cakes you can make.

Is there an easier way to solve this instead of always making a tree? YES!

Let's try to answer this using what we have learned so far.

There are 4 possible flavours for the first cake layer and then, 3 possible flavours for the second cake layer. We know the answer is...

$$4 \times 3$$

However, we can write this using factorials as well as shown below.

$$\begin{aligned} 12 = 4 \times 3 &= \frac{4 \times 3 \times \mathbf{2} \times \mathbf{1}}{\mathbf{2} \times \mathbf{1}} \\ &= \frac{4!}{2!} \\ &= \frac{4!}{(4-2)!} \end{aligned}$$

Our answer, written using factorials, is called a **permutation**. Permutations are used when we are counting objects where *ORDER DOES MATTER* and there is *no repetition*. Since we are counting how many different layered cakes we can make, the order of the flavour of the cake layers does matter and we are not repeating flavours.

Permutations

If there are n objects to choose from and we choose only k out of the n objects (where order does matter), then

$$_nP_k = \frac{n!}{(n-k)!}$$

We read $_nP_k$ as "n permute k".

We divide $n!$ by $(n-k)!$ because out of the n objects we have, we are only using k of them so we do **not** count the remaining $(n-k)$ objects.

Example Emmet is building a LEGO™ tower made of 3 blocks only. There are 12 blocks to choose from. How many different ways can Emmet build his tower?

There are 12 blocks to choose from and Emmet will use only 3 of the blocks and the order of the blocks does matter. (i.e. $n = 12$ and $k = 3$)

$$\begin{aligned} _{12}P_3 &= \frac{12!}{(12-3)!} \\ &= \frac{12!}{9!} \\ &= \frac{12 \times 11 \times 10 \times \cancel{9}!}{\cancel{9}!} \\ &= 12 \times 11 \times 10 = 1320 \text{ ways} \end{aligned}$$

Thus, there are <u>1320 ways</u> Emmet can build his LEGO™ tower.

The Birthday Problem

This is a classic counting problem and you may see many variations of this problem if you continue to study *statistics* and *probability*.

Challenge! There are 35 students in our class. What is the probability that at least two students in class share the same birthday? (Assume there are 365 days in a year.)

There are two ways of counting this:

① Find the probability of at least two students sharing a birthday. This means we have to consider the probabilities of exactly 2 students sharing a birthday, or 3 students sharing a birthday, or 4 students sharing a birthday, etc...

② Find the probability of the **complement**! The complement is the opposite event of your original event. In this case, find $P(\text{no 2 students share a birthday})$. Why can we do this? Consider the following:

$$P(\text{event happens}) + P(\text{event does not happen}) = 1$$
$$P(\text{at least 2 students share a birthday}) + P(\text{no 2 students share a birthday}) = 1$$
$$P(\text{at least 2 students share a birthday}) = 1 - P(\text{no 2 students share a birthday})$$

Yes, you can find the solution using method ① however it is faster to use method ②.
Method ②:
Since no 2 students share a birthday, everyone has a different birthday! There are 365 out of 365 possible birthdays for the 1^{st} student, 364 out of 365 possible birthdays for the 2^{nd} student, 363 out of 365 possible birthdays for the 3^{rd} student, ..., and 331 out of 365 possible birthdays for the 35^{th} student.

$$\begin{aligned} P(\text{no 2 students share a birthday}) &= \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \ldots \times \frac{331}{365} \\ &= \frac{365 \times 364 \times 363 \times \ldots \times 331}{365 \times 365 \times 365 \times \ldots \times 365} \end{aligned}$$

Take a close look at the numerator. We can write $365 \times 364 \times 363 \times \ldots \times 331$ as a permutation!

$$365 \times 364 \times 363 \times \ldots \times 331 = \frac{365 \times 364 \times 363 \times \ldots \times 331 \times 330!}{330!} = \frac{365!}{330!} = \frac{365!}{(365-35)!} = {}_{365}P_{35}$$

Now, take a look at the denominator. We are multiplying 365 to itself 35 times (for 35 students) and so we can write this using exponents.

$$\underbrace{365 \times 365 \times 365 \times 365 \times ... \times 365}_{\text{35 times}} = 365^{35}$$

Now, we have...

$$\begin{aligned}
P(\text{no 2 students share a birthday}) &= \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times ... \times \frac{331}{365} \\
&= \frac{365 \times 364 \times 363 \times ... \times 331}{365 \times 365 \times 365 \times ... \times 365} \\
&= \frac{{}_{365}P_{35}}{365^{35}}
\end{aligned}$$

Now we can calculate the probability!

$$\begin{aligned}
P(\text{at least 2 students share a birthday}) &= 1 - P(\text{no 2 students share a birthday}) \\
&= 1 - \frac{{}_{365}P_{35}}{365^{35}} \\
&\approx 0.8143
\end{aligned}$$

Thus, there is approximately an 81.43% chance that at least two students in the class share the same birthday.

Combinations (Order Does Not Matter!)

Let's use our cake example again. You're making a two-layer cake and you have four flavours: vanilla, chocolate, strawberry, and red velvet.

Question: How many different two-layer cakes can you make?
(Suppose the order of the layers does not matter.)

A table of all possible two-layer cakes has been listed below for you...

V, C	~~C, V~~	~~S, V~~	~~R, V~~
V, S	C, S	~~S, C~~	~~R, C~~
V, R	C, R	S, R	~~R, S~~

Since the order of the cake layers does not matter, we need to remove some answers from the table. For example, V, C is considered to be the same as C, V. We are currently *double counting* the possible cakes we can make.
Circle the possible two-layer cakes and cross out all the double counted cakes.

Answer: In total, there are 6 different two-layer cakes you can make.

To solve this question without all the double counting stuff, we can use something called **combinations**. We can use combinations to count objects where *ORDER DOES NOT MATTER* and there is *no repetition*.

Combinations
If there are n objects to choose from and we choose k of the n objects (where order does not matter), then

$$_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

We read $_nC_k$ or $\binom{n}{k}$ as "n choose k".

We divide $n!$ by $k!$ because there are $k!$ ways to order the selected k out of n objects and order does not matter. If we do not divide by $k!$, we are counting each combination k times and we will not have the correct solution. We also divide $n!$ by $(n-k)!$ because again, we do **not** want to count the remaining $(n-k)$ objects that were not selected.

Alright, now using combinations this time, how many different two-layer cakes can you make given that the order of the layers does not matter?

There are 4 different flavours to choose from and you are only choosing 2 flavours and the order of the cake layers does not matter. (i.e. $n = 4$ and $k = 2$)

$$\begin{aligned} {}_4C_2 &= \binom{4}{2} \\ &= \frac{4!}{2!(4-2)!} \\ &= \frac{4 \times 3 \times \cancel{2} \times \cancel{1}}{(\cancel{2} \times \cancel{1})(2 \times 1)} \\ &= \frac{4 \times 3}{2} \\ &= \frac{12}{2} \\ &= 6 \text{ different cakes} \end{aligned}$$

Thus, there are <u>6 different two-layer cakes</u> you can make.

Example 1 There are 16 students on your school's student committee and there are 3 available executive positions. How many different ways can 3 students be elected from the student committee?

We are electing 3 students from the student committee of 16 students (i.e. $n = 16$ and $k = 3$). The order in which these students are elected does not matter so we use the combination formula.

$$\begin{aligned} {}_{16}C_3 &= \binom{16}{3} \\ &= \frac{16!}{3!(16-3)!} \\ &= \frac{16 \times 15 \times 14 \times \cancel{13!}}{(3 \times 2 \times 1) \times \cancel{13!}} \\ &= \frac{3360}{6} \\ &= 560 \text{ different ways} \end{aligned}$$

So, there are <u>560 different ways</u> to elect three students from the student committee.

Example 2 Fourteen biologists applied to be part of an expedition team to study and explore the Great Barrier Reef in Australia. This team will consist of 5 selected biologists. Exactly 6 of them are trained in marine biology. If the expedition requires at least 3 of them to be trained, how many different expedition teams can be selected?

In a team of 5 biologists, at least 3 of them must be trained in marine biology. This means we have three cases: teams with 3, 4, or 5 marine biologists. For each case, we have to choose the team of marine biologists from the 6 trained and then choose the rest of the team from the 8 not trained in marine biology.

$$\begin{aligned}\binom{6}{3}\binom{8}{2}+\binom{6}{4}\binom{8}{1}+\binom{6}{5}\binom{8}{0} &= \frac{6!}{3!3!}\cdot\frac{8!}{2!6!}+\frac{6!}{4!2!}\cdot\frac{8!}{1!7!}+\frac{6!}{5!1!}\cdot\frac{8!}{0!8!}\\ &= \frac{\not{6}\times 5\times 4}{\not{6}}\cdot\frac{8\times 7}{2}+\frac{6\times 5}{2}\cdot\frac{8}{1}+\frac{6}{1}\cdot 1\\ &= 20\cdot 28+15\cdot 8+6\cdot 1\\ &= 560+120+6\\ &= 686\end{aligned}$$

There are <u>686 different expedition teams</u> that can be selected.

Pascal's Triangle

In the 16^{th} century, **Pascal's Triangle** was named after the French mathematician Blaise Pascal because of his work but interestingly enough, Pascal was definitely not the first to arrange these numbers into a triangle. It was worked on by Jia Xian in the 11^{th} century in China, then it was popularized in the 13^{th} century by Chinese mathematician, Yang Hui and became known as *Yang Hui's Triangle*. Yet again, even before Yang Hui, it was discussed and known in the 11^{th} as the *Khayyam Triangle* in Iran and was named after the Persian mathematician Omar Khayyam.

Isn't cool how people are able to study and discover mathematics from different parts of the world?

For this lesson, we will call it *Pascal's Triangle*. The triangle is built as follows:

$$\begin{array}{l}
\text{row } 0 \longrightarrow 1 \\
\text{row } 1 \longrightarrow 1 \quad 1 \\
\text{row } 2 \longrightarrow 1 \quad 2 \quad 1 \\
\text{row } 3 \longrightarrow 1 \quad 3 \quad 3 \quad 1 \\
\text{row } 4 \longrightarrow 1 \quad 4 \quad 6 \quad 4 \quad 1 \\
\text{row } 5 \rightarrow 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1
\end{array}$$

Each number in Pascal's Triangle can be written as a combination as seen below.

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}
\end{array}$$

In general, the k-th number in the n-th row can be written as $\binom{n}{k}$.

Problem Set Solutions

1. A combo meal at a restaurant includes a drink, food item, and fresh fruit. The restaurant serves two drinks (juice and pop), three food items (sandwich, wrap, and salad), and three fresh fruits (an apple, orange and grapes). How many different combos can a customer order? A customer can order $2 \times 3 \times 3 =$ 18 different combos in total.

2. D.W. goes to the library to borrow some books. There are 3 picture books, 6 fictional books, and 4 non-fictional books she is interested in reading. How many books can she read? D.W. can read $3 + 6 + 4 =$ 13 different books from the library.

3. For each of the following scenarios, state whether order matters or not:

   (a) The number of ways four distinct sets of plates at the dinner table.
   Order does matter

   (b) Mr. Elgoog is asked to draw three cards from a deck of cards. In how many ways can he select three cards? Order does NOT matter

   (c) A math student is given a list of 8 problems and is asked to solve any 5 of the problems. How many different selections can the student make?
   Order does NOT matter

   (d) Selecting a combination on a combination lock. Order does matter

4. You and a group of friends are at Canada's Wonderland planning your route around the park. You are all interested in 6 rides, 4 games, and 3 food vendors.

   (a) How many different routes consisting of one ride, one game and one food vendor could you take?

   $$6 \times 4 \times 3 = \underline{72 \text{ different routes}}$$

   (b) How different routes consisting of three rides, two games, and one food vendor could you take?

   $$6 \times 5 \times 4 \times 4 \times 3 \times 3 = {}_6P_3 \times {}_4P_2 \times {}_3P_1 = \underline{4320 \text{ different routes}}$$

5. Draw Pascal's Triangle up until the 8th row.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

6. Find the missing numbers in this row of Pascal's Triangle.

   (**Hint**: Count how many numbers there are in the row.)

   1 __13__ 78 186 __715__ 1287 1716 1287 __715__ 186 78 __13__ 1

7. Evaluate the following:

   (a) ${}_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 210$

   (b) $\binom{11}{4} = \frac{11!}{4!(11-4)!} = \frac{11!}{4!7!} = \frac{11\times10\times9\times8}{4\times3\times2\times1} = \frac{11\times10\times3\times\cancel{3}\times\cancel{4}\times\cancel{2}}{\cancel{4}\times\cancel{3}\times\cancel{2}\times1} = 330$

   (c) ${}_{15}C_2 = \frac{15!}{2!(15-2)!} = \frac{15!}{2!13!} = \frac{15\times14}{2\times1} = \frac{15\times7\times\cancel{2}}{\cancel{2}\times1} = 105$

   (d) $\frac{{}_6C_4}{{}_{12}C_3}$

   $= \binom{6}{4} \div \binom{12}{3} = \frac{6!}{4!2!} \div \frac{12!}{3!9!} = \frac{6!}{4!2!} \times \frac{3!9!}{12!}$

   $= \frac{6\times5\times\cancel{4!}\times3\times\cancel{2!}\times\cancel{9!}}{\cancel{4!}\times\cancel{2!}\times12\times11\times10\times\cancel{9!}} = \frac{\cancel{6}\times\cancel{5}\times3}{\cancel{6}\times2\times11\times\cancel{5}\times2} = \frac{3}{44}$

   (e) $\frac{{}_{14}P_5}{{}_{14}C_7}$

   $= {}_{14}P_5 \div \binom{14}{7} = \frac{14!}{9!} \div \frac{14!}{7!7!} = \frac{14!}{9!} \times \frac{7!7!}{14!} = \frac{\cancel{14!}\times\cancel{7!}\times7!}{9\times8\times\cancel{7!}\times\cancel{14!}}$

   $= \frac{7\times6\times5\times4\times3\times2\times1}{9\times8} = \frac{7\times\cancel{3}\times2\times5\times\cancel{3}\times\cancel{8}\times1}{\cancel{3}\times\cancel{3}\times\cancel{8}} = 70$

8. How many different ways can you arrange the letters of the word `TRIANGLE`?

   There are 8 different letters in the word T, R, I, A, N, G, L, and E. Therefore, there are 8! = 40 320 ways to arrange the letters.

9. A vehicle licence plate number consists of 4 letters followed by 3 digits.

   (a) How many different licence plates are possible?

   There are 26 letters and 10 digits we could use to create a licence plate. There are 26 letters to choose for the first letter, 26 letters to choose for the second letter, 26 letters to choose for the third letter, and 26 letters to choose for the fourth letter. And then there are 10 digits to choose for the first digit, 10 digits to choose for the second digit, and 10 digits to choose for the third digit. Here, we can use the Fundamental Counting Principle as follows:

   $$26 \times 26 \times 26 \times 26 \times 10 \times 10 \times 10 = \underline{456\ 976\ 000 \text{ different licence plates}}$$

   (b) How many different licence plates are possible if we cannot repeat any letters and digits?

   There is no repetition of letters and digits and the order of letters followed by digits does matter (i.e. WXYZ 123 is a different licence than ZYXW 123) and so we can use permutations to solve this question.

   $$_{26}P_4 \times {}_{10}P_3 = \underline{258\ 336\ 000 \text{ different licence plates}}$$

10. Tommy, Chuckie, Phil, Lil, and Angelica go to the movies to watch *Fantastic Beasts and Where to Find Them*.

   (a) How many different seating arrangements of the five friends are possible?

   There are 5 friends. So there are 5 friends for the first seat, 4 friends for the second seat and so on...

   $$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

   There are 120 different seating arrangments for the five friends.

   (b) Tommy and Chuckie are bros and want to sit next to each other. How many different seating arrangements are possible?

   We treat Tommy and Chuckie as one person (or a subgroup). So now we are counting the seating arrangement for 4 friends: Phil, Lil, Angelica, and Tommy & Chuckie.

   $$4! = 4 \times 3 \times 2 \times 1$$

However, we need to consider that there are $2! = 2$ ways of arranging Tommy and Chuckie's seating arrangement. (i.e. Tommy & Chuckie and Chuckie & Tommy are 2 different seating arrangements. So we have...

$$2! \times 4! = 2 \times 24 = 48$$

Thus, there are 48 different seating arrangements such that Tommy and Chuckie are sitting next to each other.

11. * How many numbers between 1000 and 9999 have only even digits?

There are only five digits that are even: 0, 2, 4, 6, and 8. Note that we are counting only four digit numbers. Any number less than 1000 is only three digits and any number greater than 9999 is a five digit number. The first digit cannot be 0 since the number would be less than 1000. This means there are four options for the first digit, and five options for every digit after that. We have...

$$\underline{\ 4\ } \times \underline{\ 5\ } \times \underline{\ 5\ } \times \underline{\ 5\ } = 500$$

Thus, there are 500 numbers between 1000 and 9999 that have only even digits.

12. * How many three-digit numbers begin with 4, 6, or 7 AND end with 0 or 1?

There 3 possibilities for the first digit (4, 6, or 7) and 10 possible digits for second digit (0, 1, 2, 3, 4, 5, 6, 7, 8, or 9) and 2 possibilities for the third digit (0 or 1) .

$$\underline{\ 3\ } \times \underline{\ 10\ } \times \underline{\ 2\ } = 60$$

There are 60 three-digit numbers that begin with 4, 6, or 7 and end with 0 or 1.

13. ** Solve for n in the following: $\binom{n}{2} = 55$

$$\begin{aligned} 55 &= \binom{n}{2} \\ 55 &= \frac{n!}{2!(n-2)!} \\ 55 &= \frac{n \times (n-1)}{2} \\ 2 \times 55 &= n \times (n-1) \\ 110 &= n \times (n-1) \end{aligned}$$

Now we need to find two consecutive factors that multiply to produce 110. After some work, you should find that $11 \times 10 = 110$.

$$110 = 11 \times 10 = 11 \times (11 - 1)$$

Therefore, $\underline{n = 11}$.

14. ** How many different ways can you order the letters of the word `MATHEMATICS`?

We have a total of 11 letters: 2 Ms, 2 As, 2 Ts, 1 H, E, I, C, and S. If all letters were considered to be distinct or different from one another, we would have 11! ways of ordering the letters. However, the letters are not distinct and we need to remove the double counted arrangements of our letters. There are 2! ways to order the two Ms, 2! ways to order the 2 As, and 2! ways to order the 2 Ts. And so, we have...

$$\frac{11!}{2!2!2!} = \frac{11!}{8} = \frac{11 \times 10 \times 9 \times \not{8} \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{\not{8}} = 4\ 989\ 600$$

Thus, there are <u>4 989 600 ways to order the letters of the word `MATHEMATICS`.</u>

15. *** Show that $\binom{11}{2} + \binom{11}{3} = \binom{12}{3}$ is true by using the combinations formula.

Let's start with the left side of the equation: $\binom{11}{2} + \binom{11}{3}$

$$\begin{aligned}
\binom{11}{2} + \binom{11}{3} &= \frac{11!}{2!(11-2)!} + \frac{11!}{3!(11-3)!} \\
&= \frac{11!}{2!9!} + \frac{11!}{3!8!} \\
&= \frac{11 \times 10}{2 \times 1} + \frac{11 \times 10 \times 9}{3 \times 2 \times 1} \\
&= \frac{11 \times 10 \times 3}{3 \times 2 \times 1} + \frac{11 \times 10 \times 9}{3 \times 2 \times 1} \\
&= \frac{330}{6} + \frac{990}{6} \\
&= \frac{1320}{6} \\
\binom{11}{2} + \binom{11}{3} &= 220
\end{aligned}$$

Now, let's check the right side of the equation: $\binom{12}{3}$

$$\begin{aligned}\binom{12}{3} &= \frac{12!}{3!(12-3)!} \\ &= \frac{12!}{3!9!} \\ &= \frac{12 \times 11 \times 10}{3 \times 2 \times 1} \\ &= \frac{1320}{6} \\ \binom{12}{3} &= 220\end{aligned}$$

We have that...

$$\binom{11}{2} + \binom{11}{3} = 220 = \binom{12}{3}$$

The equation $\binom{11}{2} + \binom{11}{3} = \binom{12}{3}$ is true.

16. *** **The Birthday Problem (continued)** There are n students in a classroom. What is the probability that at least two students in class share the same birthday?

The probability that at least two students share a birthday in a class of n students is $1 - \frac{_{365}P_n}{365^n}$.