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Continued Fractions

A Fraction of our History

Love it or hate it, there is no denying that we use fractions in our every day lives from dividing objects into desired portions, measuring weight of various materials, to calculating prices of discounted items. It may be ironic to hear that at one point in history, fractions were not even consider numbers! They were treated as a way to compare whole numbers. In fact, fractions that we use in school today were not used until the 17th century! However, fractions were a crucial first step to see that there are more just whole numbers. It would would lay the foundations for other types of numbers.

Review of Fractions

Definition 1 (**Fraction**). *A* ***fraction*** *is a number used to express how many parts of a whole we have. It is written with two numbers, one on top of each other with a horizontal line between them.*

- *The top number (**numerator**) tells you how many parts we have*
- *The bottom number (**denominator**) tells you how many parts the whole is divided into*

Definition 2 (**Equivalent Fraction**). ***Equivalent Fractions*** *are different fractions with the same value. The fractions* $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}$ *are equivalent fractions since they represent the same number*

Even though there are multiple ways to represent a fraction, we should express a fraction in it's **simplest form**. A fraction is in it's **simplest form** when no other number other than

1 can divide evenly into both the numerator and denominator.
i.e. $\frac{1}{2}$ is in reduced form, but $\frac{2}{4}, \frac{3}{6}, \frac{4}{8}$ are not.

Adding and Subtracting Fractions

1. If the fractions have a common denominator, add/subtract the numerators but **keep the denominators the same**. Proceed to step 4.
2. If they do not have a common denominator, find the **Lowest Common Multiple** (LCM) of both numbers.
3. Rewrite the fractions as equivalent fractions with the LCM as the denominator, and go back to step 1.
4. Simplify/reduce the final answer if possible

Multiplication of Fractions

1. Simplify the fractions if they are not in lowest terms. To simplify, we divide by a number that divides evenly into two numbers above, below or diagonally from each other.
$$\frac{4}{5} \times \frac{3}{8} = \frac{\cancel{4}^1}{5} \times \frac{3}{\cancel{8}^2} = \frac{1}{5} \times \frac{3}{2}$$
2. Multiply the fractions. Multiply numerator with numerator and denominator with denominator
$$\frac{1}{5} \times \frac{3}{2} = \frac{1 \times 3}{5 \times 2} = \frac{3}{10}$$

Exercise. Evaluate the following

a. $\frac{1}{5} + \frac{1}{5}$ b. $\frac{3}{4} - \frac{1}{3}$ c. $\frac{7}{16} - \frac{6}{16}$ d. $\frac{1}{4} - \frac{1}{6}$

e. $\frac{4}{8} \times \frac{1}{4}$ f. $\frac{3}{9} \times \frac{1}{7}$ g. $\frac{35}{6} \times \frac{9}{20}$ h. $\frac{2}{3} \times 4$

Division of Fractions

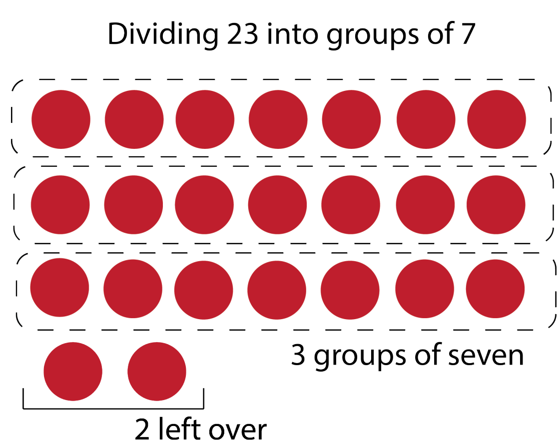
1. Take the reciprocal of the fraction following the division sign i.e. (switch the value of the numerator and denominator) and replace the division sign with a multiplication sign
$$\frac{1}{2} \div \frac{5}{3} = \frac{1}{2} \times \frac{3}{5}$$
2. Multiply the fractions as normal, remember to simplify beforehand to ease calculation

Exercise. Evaluate the following

a. $\frac{2}{3} \div \frac{7}{5}$ b. $\frac{3}{4} \div \frac{7}{4}$ c. $\frac{3}{5} \div 2$ d. $5 \div \frac{1}{4}$

The Division Statement

When we divide integers, the final answer is known as the **quotient**. When two integers don't divide evenly, we have a **remainder**. For example, let's divide 23 by 7. In other words, let's make groups of 7 from 23. The maximum group of 7s we can have is 3. That is the **quotient**. We will also have 2 left over as the **remainder**.



We can express this mathematically as:

$$23 = 7 \times 3 + 2$$

Exercise.
a) Divide 37 by 8, find the quotient, remainder and express it in this form: $37 = q \times 8 + r$

b) Divide 33 by 10, find the quotient, remainder and express it in this form: $33 = q \times 10 + r$

c) Divide 40 by 4, find the quotient, remainder, and express it in this form: $40 = q \times 4 + r$

d) Divide 127 by 23, find the quotient, remainder, and express it in this form: $127 = q \times 23 + r$

Motivation for Finding the Greatest Common Denominator

To simplify a fraction to its lowest terms, we divide by the numerator and denominator by the **greatest common divisor**.

Definition 3 (**Greatest Common Divisor**). *The largest positive integer which divides two or more integers without a remainder is called the* ***Greatest Common Divisor****, abbreviated as gcd.*

We denote the gcd of two numbers, a and b, as gcd(a, b).
e.g since the gcd of 9 and 12 is 3, we write **gcd(9, 12) = 3**.

We can use gcd(9, 12) = 3 to simplify the fraction $\frac{9}{12}$ by dividing 9 and 12 by their shared gcd of 3.

$$\frac{9 \div 3}{12 \div 3} = \frac{3}{4}$$

Exercise. Using your favourite method, determine gcd(40, 72).

However, it becomes very difficult to determine the gcd if we are given very large numbers. Consequently, fractions with large values for the numerator and the denominator are much harder to simplify.
For example:

$$\frac{52}{220}$$

Question: Is there a way to reduce or $\frac{52}{220}$ to its lowest term without trying every number and seeing if it divides both the numerator and denominator? The answer is a **resounding yes**, but we need to use something known as the **Euclidean Algorithm**. The Euclidean Algorithm can determine the **greatest common divisor**.

The Euclidean Algorithm

Example.

$$\overset{a}{37} = \underset{\text{quotient}}{4} \times \overset{b}{8} + \underset{\text{remainder}}{5}$$

Notice that:

$$\gcd(37, 8) = \gcd(8, 5)$$

Finding the gcd(8, 5) is much easier than finding gcd(37, 8) because we are now dealing with smaller numbers.

Theorem 1. *Let a, q, b, and r be positive integers in the division statement:*

$$a = q \times b + r$$

Then gcd(a, b)= gcd(b, r)

Why is this important? Notice that finding gcd(a, b) is the same as finding the gcd(b, r) but r is smaller number than a since it is the remainder. With smaller numbers, we can more easily find the gcd.

Example.
Determine the gcd(220, 52):

$$220 = 4 \times 52 + 12 \qquad \gcd(220,\ 52) = \gcd(52,\ 12)$$
$$52 = 4 \times 12 + 4 \qquad \gcd(52,\ 12) = \gcd(12,\ 4)$$
$$12 = 3 \times 4 + 0 \qquad \gcd(12,\ 4) = 4$$

Example. Determine the gcd(2322, 654) and reduce $\frac{654}{2322}$:

$$2322 = 3 \times 654 + 360 \qquad \gcd(2322,\ 654) = \gcd(654,\ 360)$$
$$654 = 1 \times 360 + 294 \qquad \gcd(654,\ 360) = \gcd(360,\ 294)$$
$$360 = 1 \times 294 + 66 \qquad \gcd(360,\ 294) = \gcd(294,\ 66)$$
$$294 = 4 \times 66 + 30 \qquad \gcd(294,\ 66) = \gcd(66,\ 30)$$
$$66 = 2 \times 30 + 6 \qquad \gcd(66,\ 30) = \gcd(30,\ 6)$$
$$30 = 5 \times 6 + 0 \qquad \gcd(30,\ 6) = 6$$

Therefore, gcd(2322, 654) = 6, and $2322 \div 6 = 387$, $654 \div 6 = 109$ and hence we have

$$\frac{654}{2322} = \frac{109}{387}$$

Exercise. Find the greatest common divisor of the following pairs of numbers using the Euclidean Algorithm:

1. gcd(45, 40) and simplify $\frac{45}{40}$ to its lowest terms
2. gcd(120, 84) and simplify $\frac{120}{84}$ to its lowest terms

Optional Section*: Finding the Lowest Common Multiple

Now suppose, we want to add two fractions with **different BUT LARGE denominators**. Finding the lowest common multiple or the common denominator may prove to be tedious. For example:

$$\frac{1}{220} + \frac{1}{52}$$

Is it possible to find the common denominator relatively quickly? YES! but let's first make another insightful observation.

gcd(a, b)	lcm(a, b)	gcd(a, b)× lcm(a, b)
gcd(2, 3)=	lcm(2, 3) =	gcd(2, 3) × lcm(2, 3) =
gcd(6, 4)=	lcm(6, 4) =	gcd(6, 4) × lcm(6, 4) =
gcd(30, 6)=	lcm(30, 6) =	gcd(30, 6) × lcm(30, 6) =
gcd(12, 20)=	lcm(12, 20) =	gcd(12, 20) × lcm(12, 20) =
gcd(28, 16)=	lcm(28, 16) =	gcd(28, 16) × lcm(28, 16) =

Can you come up with any conclusion about the product of the lcm and gcd? We arrive at a remarkably elegant relation between the lcm and gcd.

Theorem 2. *Let a and b both be positive integers, then*

$$lcm(a, b) \times gcd(a, b) = a \times b$$

Example.
Add the two fractions $\frac{1}{220} + \frac{1}{84}$

Example.
Add the two fractions $\frac{1}{1239} + \frac{1}{735}$ We have already determined gcd(1239, 735).
Using the theorem above, we get that

$$\begin{aligned} \text{gcd}(2322, 654) \times \text{lcm}(2322, 654) &= 2322 \times 654 \\ 6 \times \text{lcm}(2322, 654) &= 1518588 \\ \text{lcm}(2322, 654) &= \frac{1518588}{6} \\ \text{lcm}(2322, 654) &= 253098 \end{aligned}$$

Since the lcm(2322, 654) is 253 098. We then divide 253 098 to get multiplies of 2322 and 654 b$253098 \div 2322 = 109$ and $253098 \div 654 = 387$ Therefore, to find the common denominator, we multiply the $\frac{1}{2322}$ by 109 top and bottom and $\frac{1}{654}$ by 387 top and bottom

$$\begin{aligned}\frac{1}{2322} + \frac{1}{654} &= \frac{1 \times 109}{1239 \times 109} + \frac{1 \times 387}{654 \times 387} \\ &= \frac{109}{253098} + \frac{387}{253098} \\ &= \frac{496}{253098}\end{aligned}$$

Introduction to Continued Fractions

The problem below seems very difficult, but you actually have all the necessary math to solve it!

$$\frac{30}{7} = x + \cfrac{1}{y + \cfrac{1}{z}}$$

What is $x + y + z$ provided x, y, z are all positive integers i.e. (whole numbers) ?

$$\begin{aligned}\frac{30}{7} &= 4 + \frac{2}{7} \\ &= 4 + \cfrac{1}{\cfrac{7}{2}} \\ &= 4 + \cfrac{1}{3 + \cfrac{1}{2}}\end{aligned}$$

Observe that we took the reciprocal of a proper fraction to get an improper fraction below a numerator of 1. We did this multiple times until we had a proper fraction with 1 as the numerator. Although counter intuitive, after unwinding this expression, we can simply match the terms and conclude that $x = 4$, $y = 3$ and $z = 2$ so $x + y + z = 4 + 3 + 2 = 9$.

This type of fraction is known as a **continued fraction**.

Definition 4 (Continued Fraction). *A **continued fraction** is an expression of the form*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_n}}}}$$

Notice that the numerator (i.e the number above the fraction bar) is always 1. The other terms $\{a_1, a_2, a_3, \dots a_n\}$ are called **partial quotients**. Although this looks scary, the subscript tells us is which partial quotient we are referring to. For example, take the continued fraction below as an example:

$$2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{2}}}$$

The 2 is the first partial quotient, 3 is the second partial quotient, 4 is the third partial quotient, and the 2 is the fourth partial quotient. We write this mathematically as

$$a_1 = 2, \qquad a_2 = 3, \qquad a_3 = 4, \qquad a_4 = 2$$

Converting a Continued Fraction to an Ordinary Fraction

To convert a continued fraction to an ordinary fraction. we just simplify from the right side.

Example. Express $2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4}}}$ as a single proper fraction.

$$2 + \cfrac{1}{1 + \cfrac{1}{\frac{12}{4} + \frac{1}{4}}} = 2 + \cfrac{1}{1 + \cfrac{1}{\frac{13}{4}}} = 2 + \cfrac{1}{1 + \frac{4}{13}} = 2 + \cfrac{1}{\frac{13}{13} + \frac{4}{13}} = 2 + \cfrac{1}{\frac{17}{13}} = 2 + \frac{13}{17} = \frac{47}{17}$$

The List Notation of Continued Fractions

Writing a continued fraction is time consuming and takes up a lot of space, so there is a short hand way to write it out. Since the numerator is always 1, we only need to list the **partial quotients** in the order that they appear going down the fraction. For example, we can express the previous fraction as

$$[2; 3, 4, 2] = 2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{2}}}$$

Notice in list notation, the first number is special since it is a whole number so it is proceeded with a semi-colon; the rest of the numbers are separated by commas.

In general, the continued fraction expansion can be expressed in list notation as

$$[a_1; a_2, a_3, \ldots a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \frac{1}{a_n}}}}$$

1. Express the following as a continued fractions

a) [3; 1, 3, 5]

b) [4; 2, 3]

c) [0; 5, 9, 3]

2. Write the following continued fractions in list notation

a) $3 + \cfrac{1}{4 + \cfrac{1}{5 + \cfrac{1}{3}}}$

b) $1 + \cfrac{1}{7 + \cfrac{1}{2 + \cfrac{1}{3}}}$

We can use list notation to shorten our calculation, the trick is to notice that:

$$[a_1; ..., a_{n-1}, a_n] = \left[a_1; ..., a_{n+1} + \frac{1}{a_n}\right]$$

Example. Express [2; 1, 3, 4] as a single improper fraction

$$\begin{aligned}
&[2; 1, 3, 4] \\
&= \left[2; 1, 3 + \frac{1}{4}\right] \\
&= \left[2; 1, \frac{12}{4} + \frac{1}{4}\right] \\
&= \left[2; 1, \frac{13}{4}\right] \\
&= \left[2; 1 + \frac{4}{13}\right] \\
&= \left[2 + \frac{13}{17}\right] \\
&= \left[\frac{34}{2} + \frac{13}{17}\right] \\
&= \frac{47}{17}
\end{aligned}$$

Express the following as a single fraction.

a. [2; 1, 7]

b. [2; 2, 1, 1]

Optional Section*: Reciprocal of a Continued Fraction

Let's say we have the continued fraction expansion for $\frac{45}{16} = [2; 1, 4, 3]$ How can we easily find the continued fraction expansion for $\frac{16}{45}$ i.e. the reciprocal.

Answer: Notice that $\dfrac{16}{45}$ can be expressed as $\dfrac{1}{\frac{45}{16}}$.

$$\frac{16}{45} = 0 + \frac{1}{\frac{45}{16}}$$
$$= 0 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{3}}}}$$

To find the reciprocal of a continued fraction, we just add a 0 in front of the first partial quotient in list form.

$$\frac{45}{16} = [2; 1, 4, 3] \qquad \frac{16}{45} = [\mathbf{0}; 2, 1, 4, 3]$$

If the continued fraction already begins with a zero, then it's reciprocal is found by removing the 0 from the front of the list.

Converting an Ordinary Fraction into a Continued Fraction

There are a few important things to note.

1. **Reciprocal of a fraction**

$$\frac{1}{\left(\frac{7}{2}\right)} = 1 \div \frac{7}{2} = 1 \times \frac{2}{7} = \frac{2}{7}$$

2. **Turning a proper fraction to a mixed number/fraction** requires you find the quotient and the remainder.

$$\frac{30}{7} = 4 + \frac{2}{7}$$

Example.
Express $\frac{45}{16}$ as a continued fraction.

$$\begin{aligned}
\frac{45}{16} &= 2 + \frac{13}{16} && 45 = 2 \times 16 + 13 \\
&= 2 + \cfrac{1}{\cfrac{16}{13}} \\
&= 2 + \cfrac{1}{1 + \cfrac{3}{13}} && 16 = 1 \times 13 + 3 \\
&= 2 + \cfrac{1}{1 + \cfrac{1}{\cfrac{13}{3}}} \\
&= 2 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{3}}} && 13 = 4 \times 3 + 1
\end{aligned}$$

You may have noticed in order to convert an improper fraction to a mixed number, we need to identify the **quotient** and **remainder**. When we express it in the division statement $a = q \times b + r$, where q is the quotient and r is the remainder, it resembles the same steps we used to find the gcd in the **Euclidean Algorithm**! Not surprisingly, we can use the Euclidean Algorithm to find the partial quotients.

$$\begin{aligned}
45 &= \mathbf{2} \times 16 + 13 \\
16 &= \mathbf{1} \times 13 + 3 \\
13 &= \mathbf{4} \times 3 + 1 \\
3 &= \mathbf{3} \times 1 + 0
\end{aligned}$$

The boldfaced numbers going down are the partial quotients of the continued fraction. So we have that

$$\frac{45}{16} = \mathbf{2} + \cfrac{1}{\mathbf{1} + \cfrac{1}{\mathbf{4} + \cfrac{1}{\mathbf{3}}}}$$

Determine the continued fraction expansion of the following

a) $\frac{67}{29}$ b) $\frac{7}{10}$ c) $\frac{75}{17}$

Alain Gamache's Section with the Fibonacci Sequence

Can we express x as a number (not a continued fraction)?

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}}$$

Bees and Fibonacci Sequence Numbers. Males bees come from non-fertilized eggs, so they have a mother, but no father. Female bees from fertilized eggs, so they have a father and a mother. How many 12^{th} generation ancestors does a male bee have?

Exercise. Starting with two 1s, the Fibonacci Sequence is a list of numbers each term is the sum of two previous numbers.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

1. Convert the successive Fibonacci number ratios into continued fractions. What patterns do your answers show?

a. $\frac{1}{1}$ b. $\frac{2}{1}$ c. $\frac{3}{2}$ d. $\frac{5}{3}$

Using the pattern you observed, what do you think the continued fraction expansion is for $\frac{34}{21}$?

Ratios of Fibonacci Numbers (approximations of φ)

$$\frac{1}{1} = 1 \qquad \frac{21}{13} = 1.6153... \qquad \frac{377}{233} = 1.61802...$$

$$\frac{2}{1} = 2 \qquad \frac{34}{21} = 1.6190... \qquad \frac{610}{377} = 1.61803...$$

$$\frac{3}{2} = 1.5 \qquad \frac{55}{34} = 1.6176...$$

$$\frac{5}{3} = 1.66.. \qquad \frac{89}{55} = 1.6181...$$

$$\frac{8}{5} = 1.6 \qquad \frac{144}{89} = 1.6179...$$

$$\frac{13}{8} = 1.625 \qquad \frac{233}{144} = 1.61805...$$

If you continue this pattern, notice we have an infinite amount of 1's.

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}$$

A continued fraction doesn't necessarily need to have a set number of partial quotients. It can have an infinite number of partial quotients!!

Approximating Irrational Numbers using Continued Fractions

If you multiply an integer by itself, you get a perfect square. Below are just a few perfect squares

$$1, 4, 9, 16, 25, 36, 49, 64, 81, 100 \ldots$$

The square root of n finds the number a that when multiplied by itself it gives you n. For example, the square root of 16 is 4, since 4×4 is 16. Since we know that perfect squares are products of the same number, the square root of perfect square is an integer. However, if we square root a number other than a perfect square, we get an **irrational number**. An irrational number is a number that had an infinite decimal point.

However, if we put in $\sqrt{2}$, our calculator gives us an approximation of

$$1.41421356237$$

How does the calculator compute $\sqrt{2}$?

It may be astonishing to find that $\sqrt{2}$ can be expressed as a continued fraction! However, since $\sqrt{2}$ is irrational, the continued fraction expansion is infinite, so they have an infinite number of partial quotients. See the example below.

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}}$$

We can approximate it by cutting it off at the nth partial quotient.

$$\sqrt{2} \approx 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}$$
$$= 41/29$$
$$= 1.41379310345$$

One Last Thing: Continued Fractions don't always need to have a numerator of 1. For example, observe the number below. Can you guess what it is? Maybe you can truncate it to find out? ;D. I'm sure you know what it is!!!

$$3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{\ddots}}}}$$

Problem Set

1. An integer is divided by 6, list all possible values for the remainder.

2. A little monkey had 60 peaches.

   On the **first** day he decided to keep $\frac{3}{4}$ of his peaches. He gave the rest away. Then he ate one.

   On the **second** day he decided to keep $\frac{7}{11}$ his peaches. He gave the rest away. Then he ate one.

   On the **third** day he decided to keep $\frac{5}{9}$ of his peaches. He gave the rest away. Then he ate one.

   On the **fourth** day he decided to keep $\frac{2}{7}$ of his peaches. He gave the rest away. Then he ate one.

   On the **fifth** day he decided to keep $\frac{2}{3}$ of his peaches. He gave the rest away. Then he ate one.

   How many peaches did the monkey have left at the end.

3. Use the Euclidean Algorithm to find the gcd of the following

   a. gcd(108, 15) b. gcd(826, 624) c. gcd(114, 256)

4. Reason through to find the gcd of the following. Try a few examples. Suppose n > 1. Find

   a. gcd(n, 3n)
   b. gcd(n, n+1)
   c. gcd(n - 1, n + 1) provided that n is even
   d. gcd(n - 1, n + 1) provided that n is odd

5. Reduce the following fractions to their lowest terms

   a. $\frac{100}{150}$ b. $\frac{135}{153}$ c. $\frac{826}{624}$ d. $\frac{728}{168}$

6. In sheep talk, the only letters used are A and B. Sequence of words are formed as follows:

   - The first word only contains the letter A
   - To get the next word in the sequence, change each A in the previous word into B and each B in the previous word into AB

How many letters will there be in the 15^{th} word of sheep talk?

7. The Fibonacci sequence is a list of numbers starting with 1,1 where each term starting from the third term is the sum of the previous 2 terms. The first few terms are listed below

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144..$$

Since 2 is the third term in the list, we write it as F_3 and similarly since 21 is the 8th term, we write is as F_8. Without using the Euclidean Algorithm, determine the gcd of the following pairs of Fibonacci Numbers.

a. $\gcd(F_4, F_8)$ b. $\gcd(F_5, F_{10})$ c. $\gcd(F_6, F_{12})$

Based on the pattern above, what do you think the gcd of any two numbers of the Fibonacci sequence is?
$\gcd(F_m, F_n) =$

8. Write the following lists as continued fractions. No need to simplify!

a. [1; 2, 4, 5] b. [0; 9, 4, 3] c. [1; 7, 3, 2] d. [4; 7, 2]

9. Other than having 0 as the first partial quotient, is it possible for the other partial quotients to be 0. Why or why not?

10. Express the following fractions both as continued fractions and list notation.

i. $\frac{52}{9}$ ii. $\frac{75}{17}$ iii. $\frac{31}{264}$ iv. $\frac{67}{29}$

b. Find the reciprocal in list notation of all the fractions.

11. Express the following as one single proper fraction i.e. just one number for the numerator and the denominator.

a. $2 + \cfrac{1}{2 + \cfrac{1}{3}}$ b. $\cfrac{1}{3 + \cfrac{1}{4}}$ c. $\cfrac{1}{9 + \cfrac{1}{9 + \cfrac{1}{9}}}$

12. As a class exercise, we solved for $\frac{30}{7} = x + \cfrac{1}{y + \cfrac{1}{z}}$. Now let's suppose instead that the left hand side of the equation was equal to $\frac{8}{5}$ instead of $\frac{30}{7}$? Is there still a solution?

Can you provide another example of a simple fraction that does not equal to the right hand side of the equation?

13. Like the conventional Sudoku, this Sudoku variant consists of a grid of nine rows and nine columns divided into nine 3×3 subgrids. It has two basic rules:

- Each column, each row, and each box (3×3 subgrid) must have the numbers 1 to 9
- No column, row, or box can have two squares with the same number

The special clue-numbers in this Sudoku variant are fractions or ratios in the lowest terms.

The clue-numbers are always placed on the borderlines between selected pairs of neighboring cells of the grid. Each clue-number is the fraction of the two numbers in adjacent cells (to the left and the right). Each fraction is also written in its lowest terms, with the smaller number always denoted as the numerator. Thus $\frac{1}{2}$ can stand for the following combinations in the two adjacent cells:

1 and 2	2 and 1	2 and 4
4 and 2	3 and 6	6 and 3
4 and 8	8 and 4.