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Grade 7/8 Math Circles

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Sequences

A Sequence of Mathematical Terms

What is a sequence?

A sequence is list of objects (usually numbers) where each object has a specific position on the list.

- Objects in the list are called **terms**
- Each term is numbered so we know where to find it

{Bulbasaur, Ivysaur, Venasaur, Charmander, Charmeleon...}

Example. The Pokedex (list of all pokemon) is a sequence since each pokemon entry is numbered starting from Bulbasaur at number 1 to Volcanion at number 721. Since every entry is numbered, we know exactly where to find each pokemon. We say that the sequence is **ordered**.

We often use capital letters to indicate we are referring to a sequence. We could refer to the Pokedex, as P_n where P stands for pokedex and n is the n^{th} in the pokedex.

Since Pikachu is the 25th entry in the pokedex, we write P_{25} to show we are talking about the 25th entry in the pokedex.

Numbers in a Sequence

In mathematics, we deal with numbers as our objects. Numbers in a sequence are surrounded by curly brackets.

Example.

$$\{9, 0, 8, 3, 4\}$$

We use the letter t (for term) with a subscript (the small number) to refer to a specific number in the sequence.

$$t_1 = 9 \qquad t_2 = 0 \qquad t_3 = 8 \qquad t_4 = 3 \qquad t_5 = 4$$

i.e. 8 is the third term, so we write t to indicate we are referring to a number within the sequence and 3 as the subscript to indicate we are referring to the third number in the sequence.

Exercise.

a.	$\{1, 3, 5, 7\}$	Identify t_2 and t_3
b.	$\{5, 10, 15, 20, 25, 30\}$	Identify t_1 and t_6
c.	$\{4, 8, 16, 32, 64, 72\}$	Identify t_4 and t_5
d.	$\{1, 8, 27\}$	Identify t_3

Finite and Infinite Sequences

Sequences can have a **finite** (a certain number) or **infinite** numbers of terms. For example, the two sequences below are similar. The 1st sequence, starting at 2, jumps by 2 each term until we reach the last term of 24, whereas the other also starts at 2 and jumps by 2 each term but goes on forever. We use ... at the end of the curly bracket to let others know the sequence is infinite.

$\{2, 4, 6, 8, 10, 12, 14, ..24\}$	finite sequence
$\{2, 4, 6, 8, ...\}$	infinite sequence

Arithmetic Sequences

A sequence usually has a rule to find the value of each term.

Example. Observe the sequence below, can you find a pattern?

$$\{7, 10, 13, 16, ...\}$$

In words, we can see that we start at 7 and then add 3 to get the next term in the sequence.

This type of series is known as a **arithmetic series**, which each term differs by the same number

> **Definition 1 (Arithmetic Series).** *In an* ***arithmetic sequence*** *the difference between one term and the one right after is* ***constant***

How can we tell if a sequence is arithmetic? Take the difference of any two consecutive terms. If all the consecutive terms differ by the same number, the sequence is arithmetic.

The n^{th} term in the sequence

Suppose we are interested in finding a certain term in the sequence, just like how we might look up the 151^{th} pokemon in the pokedex.

Example. What is the 5^{th} and 70^{th} term for the sequence below?

$$\{4, 10, 16, 22, ...\}$$

We can see that any two consecutive terms differ by 6. We can add 6 to 22 to get the 5^{th} term of 28. However, this idea of repeatedly adding the constant difference may grow tiresome and tedious. We can instead derive a formula that tells us number of the n^{th} term.

The key is to notice that in order to find the 70^{th} term, we need to add 6 69 times **not seventy**!

$$\{4,\ 4+6,\ 4+\underbrace{6+6}_{2\times 6},\ 4+\underbrace{6+6+6}_{3\times 6}, ...\}$$

$$\{4,\ 4+6,\ 4+2\times 6,\ 4+3\times 6\}$$

In other words,

$$\begin{aligned} t_{70} &= 4 + 6 \times (70 - 1) \\ &= 4 + 6 \times 69 \\ &= 4 + 414 \\ &= 418 \end{aligned}$$

We can find the n^{th} term of the sequence $\{4, 10, 16, 22, ...\}$ with the formula below,

$$t_n = 4 + 6(n - 1)$$

since we are adding 6 (n-1) times to get to the nth term.

In general, we can express an arithmetic sequence like this:

$$\{t_1, t_1 + d, t_1 + 2d, t_1 + 3d, ...\}$$

We then have a formula for finding the n^{th} term in the sequence

$$t_n = t_1 + (n - 1)d$$

Where t_n is the n^{th} term in the sequence, t_1 is the first term and d is the constant difference

Example. Determine a formula for the sequence $\{5, 16, 27, 38, ...\}$ and use it to find the 10^{th} term.

Observe that we have a common difference of 11 and a first term of 5. Hence our formula is:

$$t_n = 5 + (n-1)11$$

Using our formula, we can find the 10^{th} term

$$\begin{aligned} t_{10} &= 5 + 11 \times (10 - 1) \\ &= 5 + 11 \times 9 \\ &= 5 + 99 \\ &= 104 \end{aligned}$$

Exercise: Determine an explicit formula for following sequences

a. $\{10, 17, 24, 31, ...\}$ and find the 11th term in the sequence
b. $\{7, 4, 1, -2, ...\}$ and find the 7th term in the sequence

Number of Terms in a Sequence

We can often find the number of terms between two specified numbers in a sequence.

Example. How many terms are there in the sequence $\{5, 10, 15, 20, 25, ...125\}$, how many terms are there?

After noticing the constant difference of 5 and a starting term of 5, we arrive at the formula below:

$$t_n = 5 + 5(n-1)$$

Observe that the last term in the sequence denoted by n is also the number of terms in the sequence If we sub in $t_n = 125$, we can solve for n i.e the number of terms in the sequence

$$\begin{aligned}
125 &= 5 + 5(n-1) \\
125 &= 5 + 5n - 5 \\
125 &= 5n \\
25 &= n
\end{aligned}$$

Hence we have that there are 25 terms in the sequence

Exercise. How many terms are there in the sequence $\{9, 15, 21, 27, 33, ...81\}$?

Geometric Sequences

There is another type of sequence call **geometric sequences**. Rather than adding the constant difference to get to the next term, we need multiply the previous number by a constant to get to the next term.

Definition 2 (**Geometric Series**). *A geometric sequence is where the next term is found by multiplying the previous term by a constant (called the* ***common ratio****)*

Example. Can you figure out the rule?

$$\{3, 6, 12, 24, ...\}$$

We are doubling the previous term to get to the next term in the sequence.

In this case, we say the sequence has a **common ratio**. For the specific sequence in the above example, it is 2. Moreover, we can derive an explicit formula in a manner similar to how we derived a formula for the arithmetic sequences.

$$\{3,\ 3\times 2,\ 3\times\underbrace{2\times 2}_{2^2},\ 3\times\underbrace{2\times 2\times 2}_{2^3}, ...\}$$

$$t_n = 3\times 2^{n-1}$$

In geometric series, to get from the first term to the nth term, we need to repeatedly multiply by the constant ratio (n-1) times. To shorten the amount of writing, we can use exponents.

In general, the formula for a geometric sequence is

$$t_n = t_1 \times r^{n-1}$$

Where t_1 is the first term, r is the common ratio, and t_n is the the nth term

Example. Determine the formula for the sequence $\{5, 20, 80, 320, ...\}$ and find the 6^{th} term in the sequence.

First Term:	$t_1 = 5$
Common Ratio:	4

Formula: $t_n = 5 \times t^{n-1}$

Using the geometric formula and letting n = 6:

$$\begin{aligned} t_6 &= 5 \times 4^{6-1} \\ &= 5 \times 4^5 \\ &= 5 \times 1024 \\ &= 5120 \end{aligned}$$

Exercise: Determine the formula for the following geometric sequences and find the specified n^{th} term.

a. $\{6, 18, 54, 162, ...\}$ 5^{th} term
b. $\{256, 128, 64, 32, ...\}$ 8^{th} term
c. $\{1, -1, 1, -1, 1, ...\}$ 7^{th} term

Recursive Definition

Sometimes, we may encounter sequences that are neither geometric nor arithmetic. One of the most famous sequences, the Fibonacci sequence is a prime example. Recall that the Fibonacci Sequence, starting with two 1s, is a sequence of numbers where every number starting on the 3rd term is the sum of the two numbers before it.

$$\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89...\}$$

To calculate the **third term**, we add the 1^{st} and 2^{nd} term.

$$\begin{aligned} \overset{\text{third term}}{t_3} &= \overset{\text{second term}}{t_2} + \overset{\text{first term}}{t_1} \\ &= 1+1 \\ &= 2 \end{aligned}$$

To calculate the **fourth term**, we add the 3^{rd} term and 2^{nd} term.

$$\begin{aligned} \overset{\text{fourth term}}{t_4} &= \overset{\text{third term}}{t_3} + \overset{\text{second term}}{t_2} \\ &= 1+2 \\ &= 3 \end{aligned}$$

Similarly, to calculate the **fifth term**, we add the 4^{th} term and 3^{th} term.

$$\begin{aligned} \overset{\text{fourth term}}{t_5} &= \overset{\text{third term}}{t_4} + \overset{\text{second term}}{t_3} \\ &= 2+3 \\ &= 5 \end{aligned}$$

In general, to calculate the n^{th} term of the Fibonacci sequence, we add the $(n-1)^{\text{th}}$ and $(n-2)^{\text{th}}$.

We can write this as

$$\underset{\text{current term}}{t_n} = \underset{\text{one term back}}{t_{n-1}} + \underset{\text{two terms back}}{t_{n-2}}$$

We call this a **recursive sequence**. In recursion/recursive sequence, we use the previous term to calculate the next term. When we write sequences in a recursive manner, we need an initial term/number, otherwise we have no initial term to use to get to the next term.

Example. Given that $t_1 = 3$. Find the next 2 terms with the given recursive formula.

$$\underset{\text{current term}}{t_n} = 2 \underset{\text{previous term}}{t_{n-1}} + 1$$

Since we already have 3 as our first term, the only thing to do is determine the second term.

Using $n = 2$, we have

$$\begin{aligned} t_2 &= 2 \times t_{2-1} + 1 \\ &= 2 \times t_1 + 1 \\ &= 2 \times 3 + 1 \\ &= 6 + 1 \\ &= 7 \end{aligned}$$

Using $n = 3$, we have

$$\begin{aligned} t_3 &= 3 \times t_{3-1} + 1 \\ &= 3 \times t_2 + 1 \\ &= 3 \times 7 + 1 \\ &= 21 + 2 \\ &= 23 \end{aligned}$$

Exercise.

1. Calculate the 5^{th} term for the recursive sequence $F_n = F_{n-1} + F_{n-2}$, with $F_1 = 1$ and $F_2 = 1$

2. Calculate the 4^{th} term for the recursive sequence $S_n = 3S_{n-1} + 2S_{n-2}$ with $S_1 = 2$ and $S_2 = 5$

The Babylon Method: Approximating Square Roots with Sequences

The ancient Babylonians used recursive sequences to approximate square roots. Recall that the square root of any number other than a perfect square results in an irrational number (infinite non-repeating decimal number). However, most of the time, it is sufficient for us to approximate a decimal number by finding the first few decimal places.

Example. $\sqrt{3}$ is irrational, but when we type $\sqrt{3}$ into our calculators, it gives us the first 9 decimal places

$$1.732050806$$

The method we will be seeing in this Math Circles is called the Babylon Method where each term in the sequence is a better approximation of square root than the previous term.

Suppose we want to find the square root of 3. We will use 3 as our initial term. The following terms can be found using the formula below:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{3}{x_n}\right)$$

$$
\begin{aligned}
x_2 &= \frac{1}{2}\left(3 + \frac{3}{3}\right) \\
&= \frac{1}{2}(3+1) \\
&= \frac{4}{2} \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= \frac{1}{2}\left(2 + \frac{3}{2}\right) \\
&= \frac{1}{2}\left(\frac{4}{2} + \frac{3}{2}\right) \\
&= \frac{1}{2} \times \frac{7}{2} \\
&= \frac{7}{4} \\
&= 1.75
\end{aligned}
$$

$$
\begin{aligned}
x_4 &= \frac{1}{2}\left(1.75 + \frac{3}{1.75}\right) \\
&= \frac{1}{2}(1.75 + 1.714285714) \\
&= \frac{1}{2}(3.464285714) \\
&= 1.732142857
\end{aligned}
$$

$$
\begin{aligned}
x_5 &= \frac{1}{2}\left(1.732142857 + \frac{3}{1.732142857}\right) \\
&= \frac{1}{2}(1.732142857 + 1.731958763) \\
&= \frac{1}{2}(3.46410162) \\
&= 1.73205081
\end{aligned}
$$

You can see that by the fourth term, we get something very very close to what the calculator gives us.

In general, we can find the square root of S, with the following recursive formula:

$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{S}{x_{n-1}}\right)$$

with $x_1 = S$.

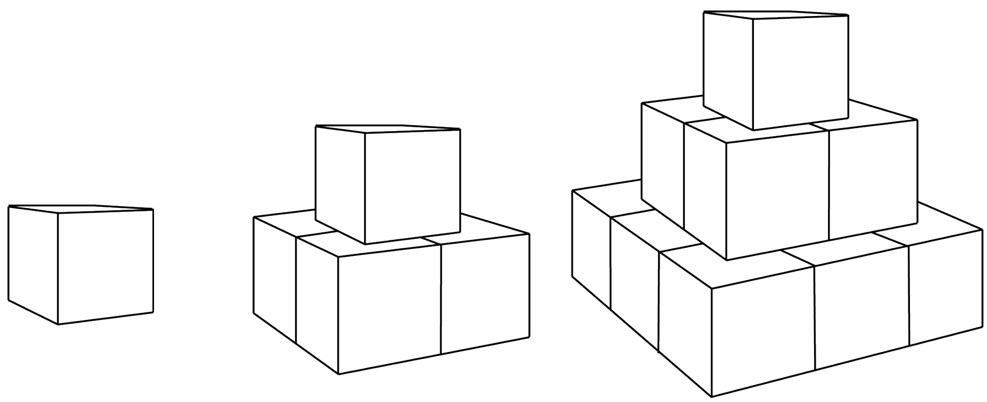
Exercise. Using the Babylonian Method calculate $\sqrt{5}$ up to the fifth term. Then check with your calculator.

Problem Set

1. Determine if the following sequences are geometric, arithmetic, or neither.
   (a) $\{4, 11, 30, 67, ...\}$
   (b) $\{-15, -12, -9, -6...\}$
   (c) $\{2, 6, 18, 54, ...\}$
   (d) $\{\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, ...\}$
   (e) $\{1, 0, 1, 0, 1, ...\}$
2. Which term in the sequence {2, 5, 8, 11, ...} is equal to 83?
3. Determine the first 3 terms of the following sequences
   (a) $t_n = 2n + 5$
   (b) $t_n = n^2 + 1$
   (c) $t_n = 3 - n$
4. Find the fourth term in the following recursive sequences
   (a) $t_n = 5t_{n-1}$ $\qquad t_1 = 4$
   (b) $t_n = 2t_{n-1} + 2t_{n-2},$ $\qquad t_1 = 1,\ t_2 = 3$
5. Given the sequence $\{5, 9, 13, 17, ...\}$, is 413 in the sequence? If so what term does it correspond to?
6. Determine the value of x such that $x + 2, 2x + 3, 4x - 3$ are consecutive term of an **arithmetic** sequence.
7. Determine t_{20} in the arithmetic sequence where $t_6 = 28$ and $t_{11} = 63$
8. A ball is dropped from a height of 60 m. The ball bounces $\dfrac{4}{5}$ of the distance it falls.
   (a) To what height does the ball bounce after the first bounce
   (b) To what height, to one decimal place, does the ball bounce after the third bounce
9. Using the Babylon method, approximate $\sqrt{6}$ up to 4 iterations. Then check your answer with a calculator

10. Does the Babylon method work with perfect squares i.e $\sqrt{4}$. What happens when you use the Babylon method to approximate perfect squares. Try a few examples.

11. The ancients built square-based structures similar to those shown in the diagram. They began with 1000 identical cubes and wished to build as many structures as possible. The first structure contained in two layers. Beside it was constructed a second layer with three layers. This process was continued as shown until the number of cubes left was not sufficient to build the next structure. How many cubes were left.



12. Find the last digit in the 47th term in the sequence $\{7^1, 7^2, 7^3, 7^4, ...\}$ without using a calculator.

13. We have the following formula below to calculates different terms of a certain sequence. Try for n = 1, 2, 3, 4, 5. What do you see?

$$t_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$