UNIVERSITY OF WATERLOO

Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

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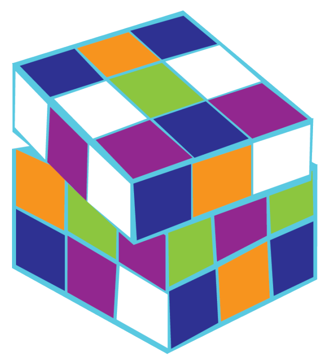
Visual Group Theory

Grouping Concepts Together

We will start with one of the famous toys in history, the Rubik's cube, to explore a new branch of mathematics. Invented in 1974 by Erno Rubik of Budapest, Hungary, the Rubik's cube comes prepackaged in a solved position, where each face of the cube has the same colour. However, we can scramble the cube by rotating any one of it's six faces. The goal of this particular puzzle is to return the cube back to it's original/solved position.

The Rubik's cube is of significant mathematical interest because of it's symmetrical nature. Symmetry is present everywhere in mathematics, but nowhere as studied or observed than in **Group Theory**. Can you give or think of examples of symmetry?



What is group theory?

We will use three key observations from the Rubik's cube that are of significant interest to us.

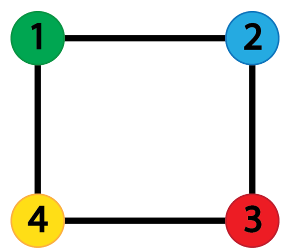
In the Rubik's cube,

- There are a set of actions you perform on the cube i.e. you can rotate any of it's 6 sides
- Each action can be reversed i.e. you can rotate the other way to cancel out your initial rotation
- Combining actions results in another action

Using this we will make the Math Circles definition of what a **group** is.

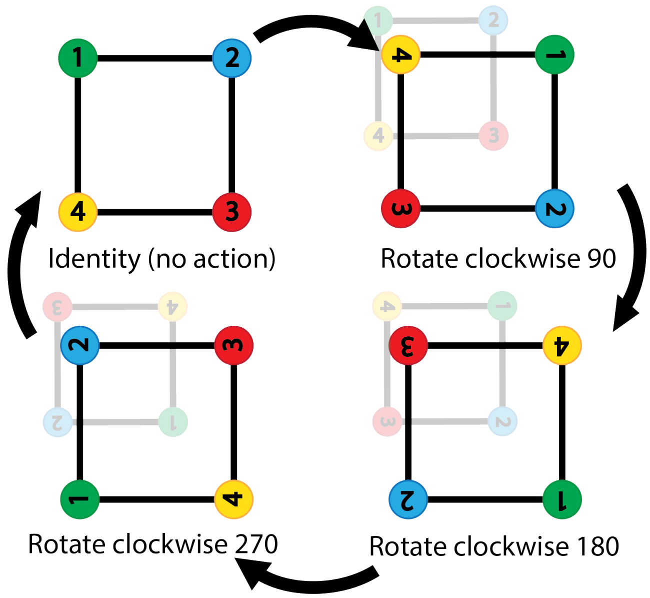
Definition 1 (Group). There is a list of predefined actions
(Inverse Element) Every action is reversible by another action
(Identity) You are allowed to do nothing
(Closure) Any sequence of consecutive actions results in an action we previously allowed

Example 1. (Rotations)
Imagine you are given the square below, with the numbers labeled 1,2,3,4 on the corners of the square and you are ONLY allowed to rotate it **clockwise** by 90°, 180°, and 270°.



We can see how this is a group.

- We have **4 allowable actions-** do nothing, rotate 90° clockwise, 180° clockwise, and 270° clockwise
- A rotation can be undone by another rotation. For example, if I rotate 90°, and then I rotate 270°, I'll return the square back to it's original position
- Two rotations combined is equivalent to another rotation.



To simplify the notation, we will the use the following to represent our actions

$$\{I, R_{90^\circ}, R_{180^\circ}, R_{270^\circ}\}$$

We can use two actions to formulate a third action. For example, combining a rotation of 90 degrees and 180 degrees gives me a rotation of 270. We will write it as

$$R_{90^\circ}R_{180^\circ} = R_{270^\circ}$$

Exercise.

1. What is $R_{90^\circ}R_{90^\circ}$? Draw out how the square looks like after the two rotations.
2. What is $R_{180^\circ}R_{270^\circ}$? Draw out how the square looks like after the two rotations.
3. If I rotated 270° 53 times, what will my square look like at the end?
4. When we combine two rotations we always end up with another rotation. Does the order how you combine the rotation matter? For example, if I rotated 90° clockwise and then 180° is it the same as rotating 180° clockwise and then 90° clockwise? Justify your answer!

Example 2. (A Non-Group)

If we are not careful with the actions we allow, it may not be a group! Using the same square, let's say we are only allowed two actions - **flipping vertically** and **flipping horizontally**. You may also assume we can do nothing as well. Let's denote them as f_v for flipping vertically and f_h as flipping horizontally. Is this a group?

If it's not a group, can we add an action to fix this?

Hint: Remember that two actions must combine to from our list of allowed actions. It may be helpful to draw out every combination of the two actions.

How might we fix this then?

We can add the action of:
In total our four actions are now:

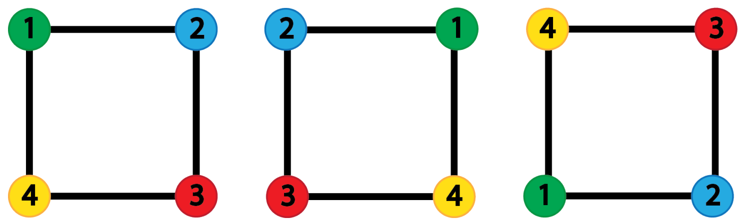
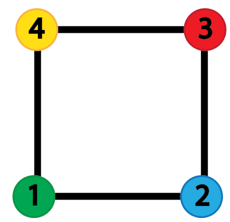
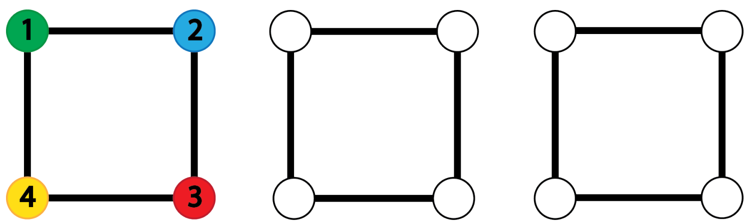
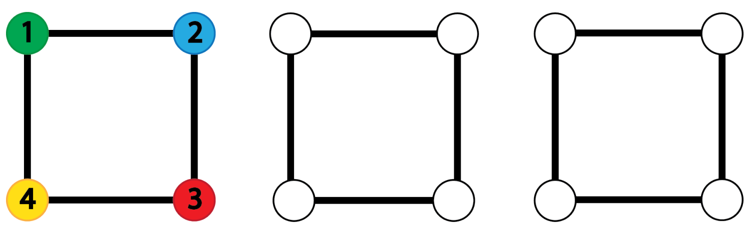
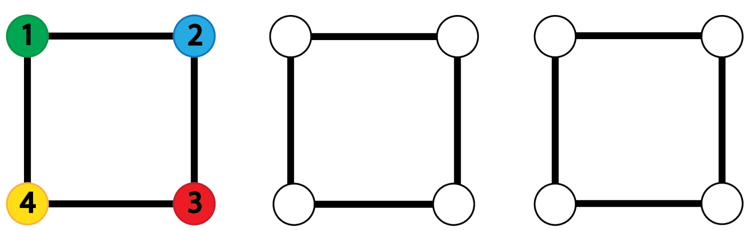
Notice that reflecting horizontally and vertically is the equivalent as rotating the square 180°. So perhaps, we can just add a rotation of 180° as an action to our list, but we must check that when we combine a rotation of 180° with either a horizontal or vertical reflection, we get back one of our actions in our list.

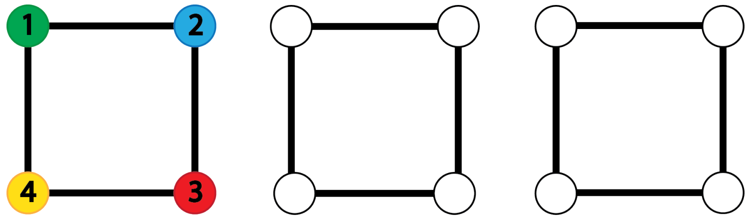
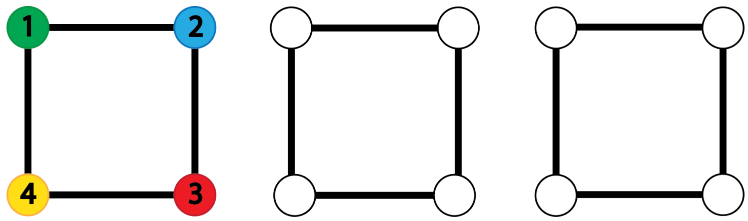
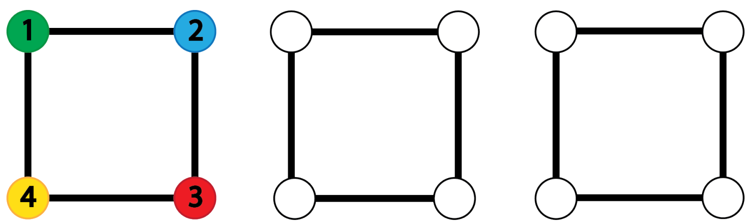
So far we have four actions

$$\{F_v, F_h, R_{180^\circ}, I\}$$

Let's try out the various possibilities.

In the table below, notice that the vertical flip followed by a rotation by 180° is the same a horizontal flip. Now try out the remaining combinations of rotations and reflections to see if it returns a previously allow action.

Combination of Action	**Equivalent Action**
$F_v R_{180^\circ}$	F_h
Reflect Vertically Rotate 180 	Reflect Horizontally 
$R_{180^\circ} F_v$	
	
$F_h R_{180^\circ}$	
	
$R_{180^\circ} F_h$	
	

F_vF_v	
	
F_hF_h	
	
$R_{180^\circ}R_{180^\circ}$	
	

1. After checking every combination of our allowed actions i.e I, F_v, F_h and R_{180°, does it always result in an action that is in our list. Can we declare that with the addition of rotating 180°, that we now have a group?
2. Is every action reversible? How is this different from the 1st example with rotations? How are the actions that reverse rotations different from those from reflections?

Organizing Group Actions: Cayley Tables

Drawing every possible combination of our permitted actions quickly becomes cumbersome. Instead, we can construct a square table to see all the possible combinations of actions performed on a square. This is called a **Cayley Table**.

Example.

Going back to our first example with the rotations. We can express all combinations succinctly the chart shown below.

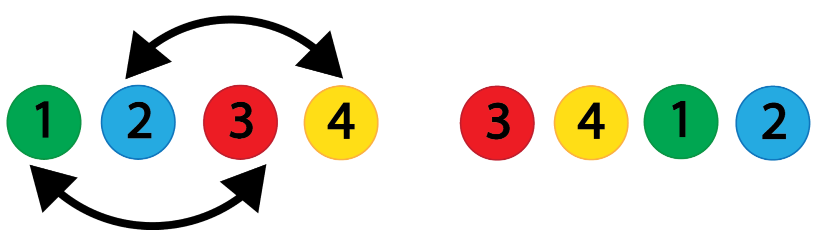
Action	**I**	R_{90°	R_{180°	R_{270°
I				
R_{90°				
R_{180°				
R_{270°				

Exercise.

Construct the Cayley Table for Example 2 with our 3 actions in addition to doing nothing - Rotation Clockwise 180°, Vertical Reflection, and Horizontal Reflection.

The Action of Swapping Places: Permutation Groups

Now let's another type of action we can do - rearranging the order of 4 balls. We call different rearrangements - **permutations**. To rearrange or to permute the order of our objects, we may swap the location of any two objects. For example, we have four balls, let's swap the 2nd ball's location with the 4th ball, and the 3rd ball with the 1st ball's location.



Question: Does swapping the location of objects, a group?

List all the possible different ways, you can arrange the 4 balls shown above.

Hint: It may be helpful to determine the total number of different arrangements first.

Example.

Suppose I have the 4 balls lined up from 1 to 4 in order. Instead of swapping, let's relocate each ball to a different position.

- I move the ball from the **first position** to the **fourth position**
- I move the ball from the **second position** to the **first position**
- I move the ball from the **third position** to the **second position**
- I move the ball from the **fourth position** to the **third position**

What does my final arrangement look like?

We write this mathematically as

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

We shall call this the **rearrangement rule**

Here we have an array of numbers, where the top row indicates the which position we are referring to initially, and the bottom number indicates which position we are sending the ball. For example, below the number 1 on the top row is 4. The ball that is located in position 1 is now placed in the fourth position. Similarly the ball in position 2 on it is now placed in the first position and so forth.

Here is something more interesting, let's move every ball from a position **twice**. Suppose I have 4 balls as shown below again.

- I move the ball from the **first position** to the **second position**
- I move the ball from the **second position** to the **third position**
- I move the ball from the **third position** to the **fourth position**
- I move the ball from the **fourth position** to the **first position**

Now with the balls already moved once from their initial position. Let's move them again.

- I move the ball form the **first position** to the **third position**
- I move the ball from the **second position** to the **fourth position**
- I move the ball from the **third position** to the **second position**
- I move the ball from the **fourth position** to the **first position**

We write this mathematically as:

$$\overbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}}^{\text{Second Rearrangment}} \overbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}}^{\text{First Rearrangement}}$$

When we combine permutations, we read from **right to left**. What does the final configuration look like?

Exercise.

For the following rearrangement actions, determine the equivalent action. and draw the final configurations of where the balls are.

1. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$
2. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 2 & 3 \end{pmatrix}$

Undoing the Rearrangement

Suppose we are given the rearrangement rule in the array below, how can return all the balls back to it's original position?

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

1. Can you create another rearrangement rule that returns all the balls to their initial position?

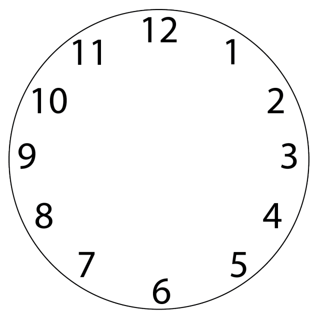
2. Is it possible to keep applying the same rearrangement rule to return all the balls to their initial position?

The Futurama Problem

An episode of Futurama, the prisoner of Brenda, received critical acclaim for popularizing math. In this episode, Professor Farnsworth and Amy build a machine that allows them to switch minds. However, the machine can only switch minds between two bodies only once, so they are unable to return to their bodies. In an attempt to return to their original bodies, they can invite other people to switch bodies with them. Is it possible for everybody to return to their original body? If so, how can this be done? How many people do they need to invite?

Problem Set

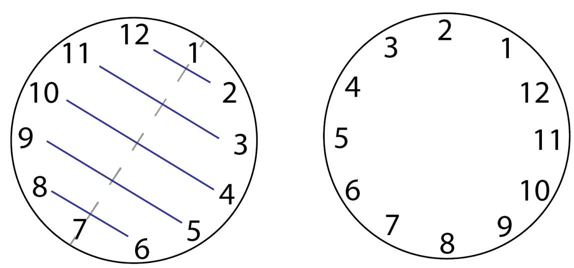
1. **Clock**. The clock is an interesting source of symmetry which naturally makes it of mathematical interest.



   (a) Suppose we can only rotate the clock by 1 hour. How many possible rotations are there?

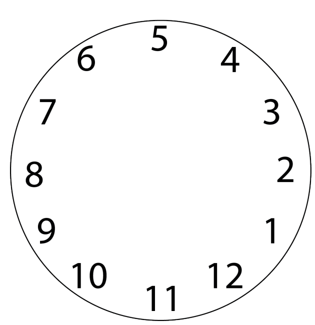
   (b) How many possible reflections are there? A reflection is done by drawing between two numbers on a clock diametrically opposite away from each other (equal distance away from other). For example 12 and 16 are diametrically opposite as well as 10 and 4. Then all the number reflect across that line.

   (c) If I combine to reflections together, what is their equivalent action?



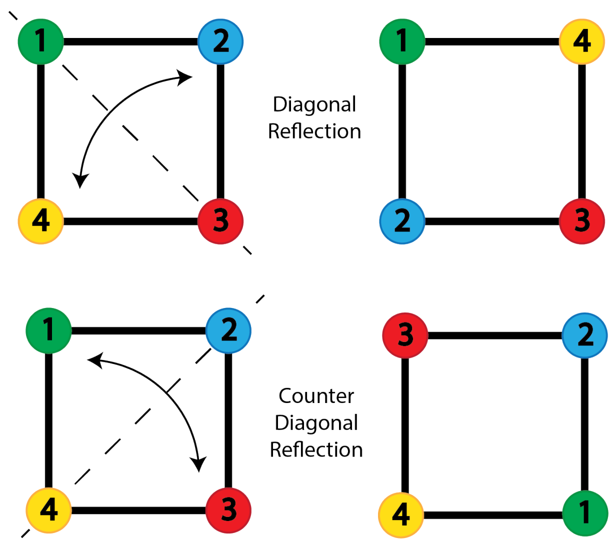
7 and 1 are diametrically opposite so the line from 7 to 1 is the line of reflection
blue line indicates which numbers are swapped

   (d) The clock below is scrambled. Can you using just rotations and reflections, return the clock back to it's normal face? How many actions do you require? Can you come up with multiple ways?



2. **The Light Switch**. Suppose we have two light switches one next to the other. You have the following actions - flipping the first switch, flipping the second switch, switching both switches, and as usual doing nothing. Draw all the possible configurations. Is this a group?

3. Using the square below (the same as the class example), but now we add a reflection diagonally



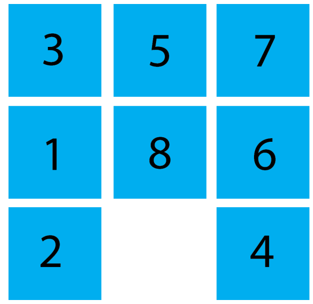
With the addition of these two actions (reflection diagonally) F_d and a reflection counter diagonally F_c, along side the actions we did in class i.e. rotate by 90° $R_{90°}$, rotate by 180° $R_{180°}$, rotate by 270° $R_{270°}$, horizontal reflection F_h, and vertical reflection F_v. Draw out the Cayley Table. After seeing the Cayley Table, determine if this is a group.

4. Simplify multiple permutation actions as one equivalent permutation action and draw out the final configuration.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

5. A coach must choose 5 players out of 12 in tryout to put on the curling team. How many possible ways can they choose those 5 players

6. **Sliding Puzzle** In the sliding puzzle, there is vacant spot, you may move any block adjacent to the vacant space (either horizontally or vertically in the vacant spot). Is it possible to keep moving these places around given one space to arrange the 3×3 block into block that puts all the number in order.



7. **Three Cups Problem** We are given three cups. One cup is upside down, and the other two is right-side up. The objective is to turn all cups right-side up in no more than six moves. Each time, you must turn over exactly two cups per move. Is this possible?

8. The director of a prison offers 100 death row prisoners, who are numbered from 1 to 100, a last chance. A room contains a cupboard with 100 drawers. The director randomly puts one prisoner's number in each closed drawer. The prisoners enter the room, one after another. Each prisoner may open and look into 50 drawers in any order. The drawers are closed again afterwards. If, during this search, every prisoner finds his number in one of the drawers, all prisoners are pardoned. If just one prisoner does not find his number, all prisoners die. Before the first prisoner enters the room, the prisoners may discuss strategybut may not communicate once the first prisoner enters to look in the drawers. What is the prisoners' best strategy?