UNIVERSITY OF
WATERLOO

Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

October $4^{th}/5^{th}$

Mathematical Puzzles

Mathematical Reasoning

To many people, it may be surprising to hear that mathematics is not about adding huge numbers and formulating large equations. Math is more of discovering patterns, making connections, and verifying truth than it is about adding big numbers in the shortest time possible. In many ways, mathematics is like a puzzle; you must be able to put all the pieces together. This requires problem solving skills and a fun to way to develop these skills is to solve mathematical puzzles. These puzzles retain the same spirit and soul of mathematics but without the formalism shown in school. Since each problem is different, it can be frustrating but most of time, it can be a rewarding and deeply enriching experience when you crack the puzzle.

Problem Solving Guidelines

1. Read the question multiple times. The first time, you should read to understand what the question is asking. Then read again to identify missing or important information.

2. Draw and/or sketch diagrams, charts, picture, or other visuals. This helps organize your thought process and point out important information quickly

3. If a problem seems too difficult to solve all at once. Break it down to smaller, more manageable steps. It is then easier to piece them together for a final solution

4. Identify any patterns you may see. Ask questions like "Am I seeing something happen again and again?"

5. It may be helpful to eliminate any impossible (but not **UNLIKELY**) solutions or answers

6. Work backwards. Can I start at the end of the question to help me figure it out? What might possible solutions look like?

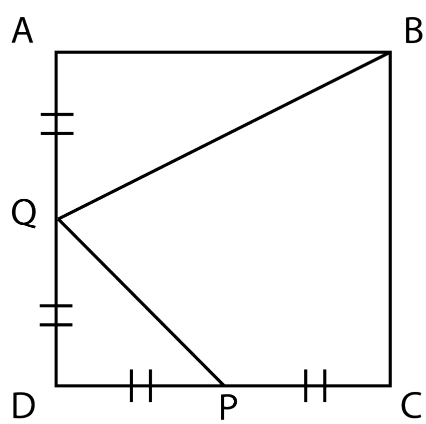
7. Sometimes you may have to make a conjecture (a reasonable guess) of what rules they follow and be able to verify if your initial guess was indeed correct.

1. The Birthday Cake

You must cut a birthday cake into exactly eight pieces, but you are only allowed 3 straight cuts. How can this be done? Draw your solution below.

2. The Square Area

In square ABCD, P is the midpoint of DC and Q is the midpoint of AD. If the area of the quadrilateral QBCP is 15, what is the area of square ABCD?

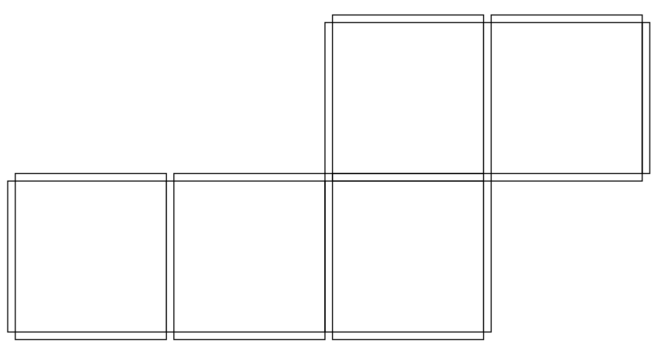


3. The Insurance Salesman

An insurance salesman walk up to house and knocks on the door. A woman answers, and he asks her how many children she has and how old they are. She says she has 3 children. If you multiply the their ages, you get 36. He says this is not enough information. So she gives him a 2nd hint. If you add up their their ages, the sum is the number on the house next door. He goes next door and looks at the house number and says this is still not enough information. So she says she'll give him one last hint which is that her eldest of the 3 plays piano.

4. From 5 squares to 4 Squares

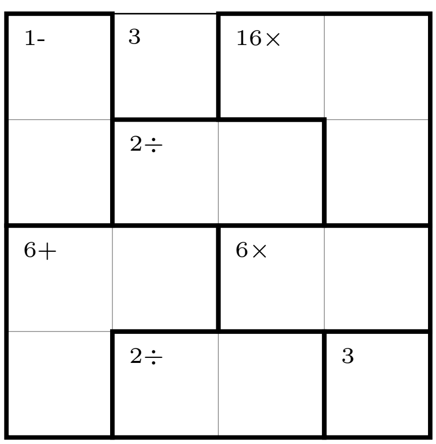
Below you see are 5 squares. Move **two and only two toothpicks** to form 4 squares each with a side length of 1 toothpick. No other shapes are permitted.

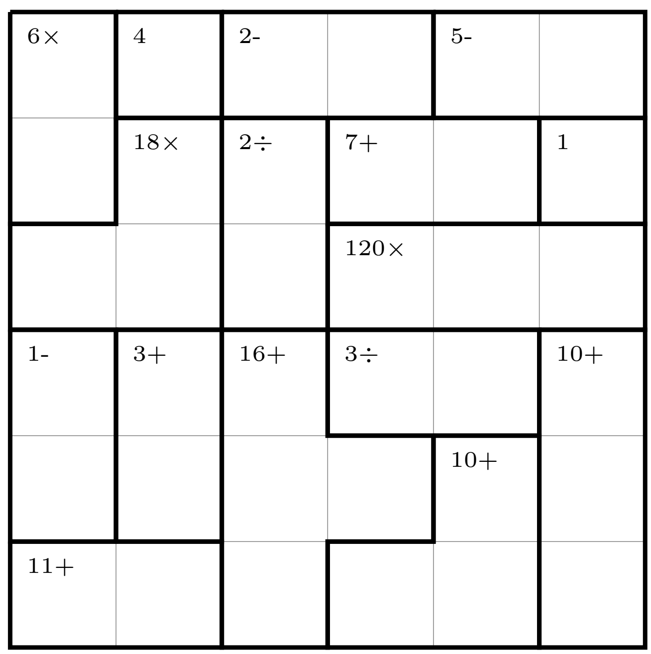


5. KenKen Puzzles

In KenKen, the goal is to fill the whole grid with numbers but it must satisfy the following conditions

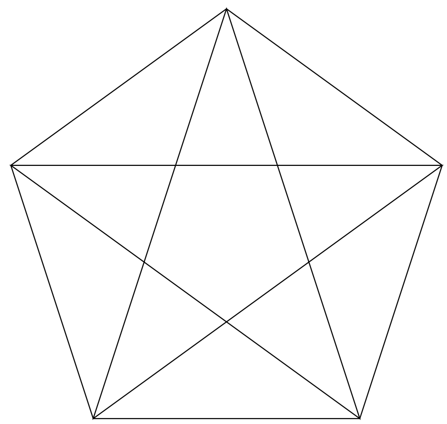
1. In a grid with side length of n, you may only use numbers from 1 to n. For example if you had a 4 by 4 grid, you may only use numbers 1-4
2. No number can be repeated in any row or column
3. Areas with bolded outlines are called “cages”. The top left corner of each cage has a target number and an operation. Numbers in the cage must combine (in any order) using the given mathematical operation (+, -, × ÷) to produce the target number





6. Counting Triangles

In the pentagon below, how many different triangles are there?



7. Divisibility

A teacher wrote a large number on the board and asked students to determine what are the divisors of this number.

The first student said, “the number is divisible by 2.”
The second student said, “the number is divisible by 3.”
the third student said, “the number is divisible by 4.”

$$\vdots$$

The 30th students said, “the number is divisible by 31”

The teacher then said that exactly two students, who spoke consecutively were incorrect. Which students spoke incorrectly?

8. A Big Number

Using the digits from 1 to 9, two numbers must be made. The product of these two numbers should be as large as possible. What are these numbers?

9. The Potato Paradox

You have 100 lbs of potatoes, which are 99% water by weight. You let them dehydrate until they are 98% Water. How much do they weigh now?

Hint: Take your time, the answer for most people is rather surprising!

10. The Circular Track

On a circular track, Alphonse is at point A and Beryl is diametrically opposite at point B. Alphonse runs counterclockwise and Beryl runs clockwise. They run at constant, but different, speeds. After running for a while they notice that when they pass each other it is always at the same three places on the track. What is the ratio of their speeds?

11. Bridge Crossing

A group of four people has to cross a bridge. It is dark and the only possible way to cross the bridge safely is to light the path with a flashlight. Unfortunately, the group has only one flashlight. Only two people can cross the bridge at the same and different members of the group takes a different time to cross the bridge.

- Alice crosses the bridge in 1 minute
- Bob crosses the bridge in 2 minutes
- Charlie crosses the bridge in 5 minutes
- Dorothy crosses the bridge in 10 minutes

How can the group cross the bridge in the minimum amount of time possible?

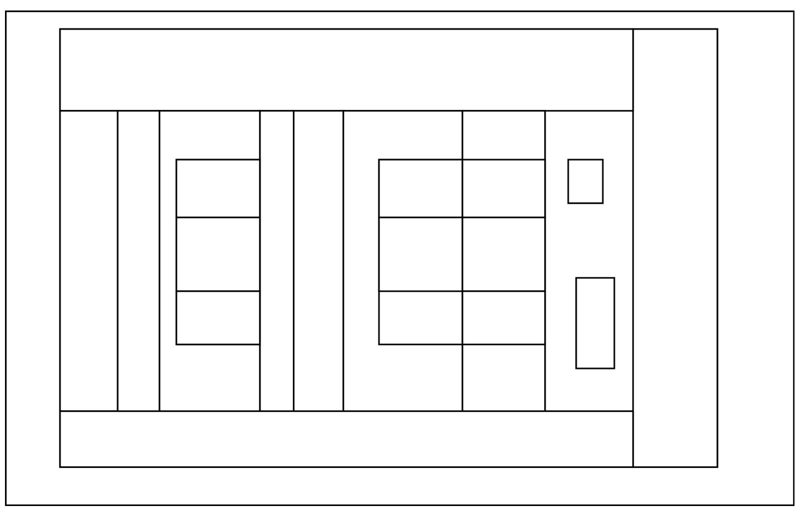
12. Four Colouring Theorem

To create maps that are easy to read, mapmakers often color them according to a rule that touching regions must always be colored differently. To color a large, complicated map this way, you might think you'd need to use a lot of different colors. But in fact, it has been proven mathematically that you never need more than four colors, no matter what the map looks like. Francis Guthrie made this conjecture in 1852, but it remained unproven until 1976, when Wolfgang Haken and Kenneth Appel showed that it was true!

Also, quite interestingly, this proof required the assistance of a computer to check 1,936 different cases that every other case can be reduced to! To date no one knows a quick short proof of this theorem.

The drawings below aren't maps, but the same principle applies to them. Can you find a way to color all the regions in each drawing, using no more than four different colors, so that regions of the same color never touch (except at corners)?

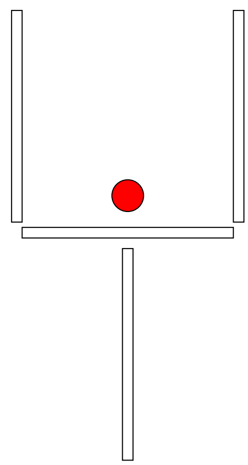
Hint: Before coloring a pattern, plan how you will do it by penciling in the names of your chosen colors in each region.



Problem Set

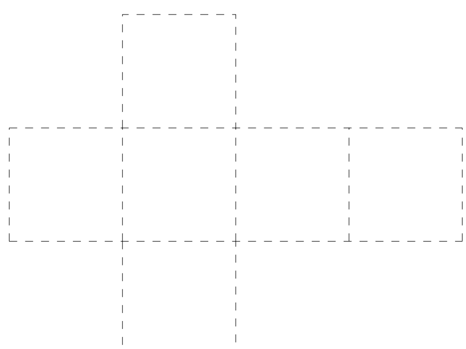
1. The Cherry in the Glass

In the image below, we have a cherry in a glass bottle. The glass bottle is composed of 4 sticks. Can you, by moving 2 matchsticks only and not touching the cherry, re-make the glass with the cherry outside?

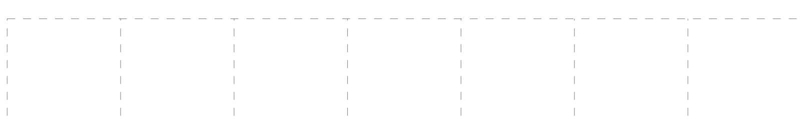


2. From Strip to Cube

In school, you have folded two-dimensional sheets of paper called nets to form 3-D shapes. Traditionally, a cube can be folded by the net shown below.

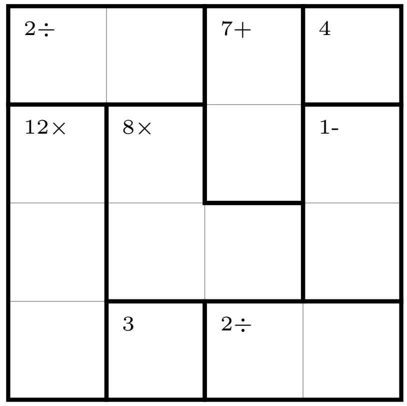


It turns out that you can fold a cube from a strip of paper as well! The minimum strip that can be folded into a cube is one unit wide and seven units long (1×7). Show how can this be done.



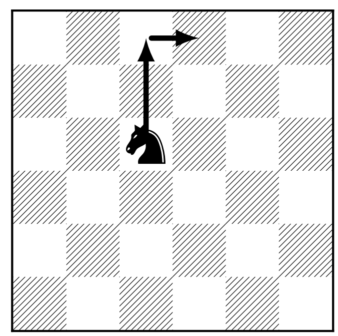
3. More Ken Ken Puzzles

Solve the Ken Ken Puzzle below.



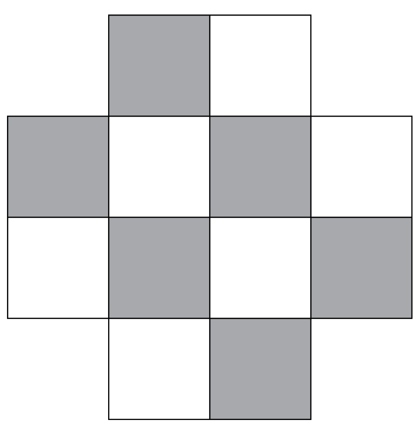
4. The Knight's Tour

One of the most well-known math problems, historically, is the "The Knight's Tour". In Chess, the knight can only move in an L-shape as shown on diagram on the right.



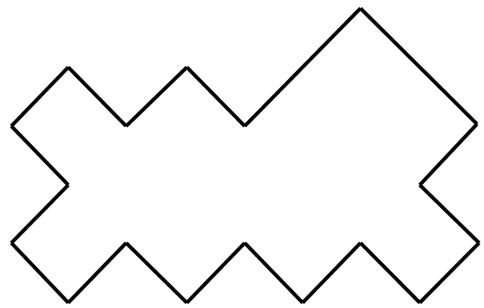
Starting from any square on the chessboard, the objective of the knight's tour is to visit every square every on the board ending at the knight's initial square on the last move without landing on the same square more than once. We investigate knight's tour problem on a modified chess board as the usual 8 x 8 block is beyond the scope of Math Circles.

Find the path the knight on the board below.



5. Crazy Cut

Divide the figure below into two using one cut.



6. Missing Dollar Riddle

This is a well known problem that highlights fallacies behind human reasoning.

Three people check into a hotel. The clerk says that the bill is $30, so each guest pays $10. Later the clerk realizes the bill should only be $25. The clerk plans on returning $5 to the guest, but on his way to their room, he realizes that he can not divide $5 equally into 3. Since the guests were unaware of the revised bill, he decides to give each guest $1 back and keep the remaining $2 for himself as tip. Each guest got $1 back, so now each guest paid only $9 with a total amount of $27 among the three of them. The clerk has $2. However, if we add the total amount paid with the amount clerk kept for himself, we have $27 + $2 = $29. If the guests originally paid $30, what happened to the remaining $1?

7. Notable Number

There is a notable number of ten digits, with the following properties **starting from the right**:

All digits from 0 to 9 appear only once in this number
The first digit is divisible by 1
The number formed by the first 2 digits is divisible by 2
The number formed by the first three digits is divisible by 3
The number formed by the first four digits is divisible by 4
The number formed by the first five digits is divisible by 5
The number formed by the first six digits is divisible by 6
The number formed by the first seven digits is divisible by 7
The number formed by the first eight digits is divisible by 8
The number formed by the first nine digits is divisible by 9
The number formed by the ten digits is divisible by 10.

8. The Sum and the Product

A teacher says: "I am thinking of two natural numbers bigger than 1. Try to guess what they are.""

The first student knows their product and the second student know their sum.

First:	"I do not know the sum."
Second:	"I knew that. The sum is less than 14."
First:	"I knew that. However, now I already know the numbers."
Second:	"And so do I."

What were the two numbers?