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Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

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Mathematical Puzzles and Recreational Mathematics

Mathematical Reasoning

To many people, it may be surprising to hear that mathematics is not about adding huge numbers and formulating large equations. Math is more of discovering patterns, making connections, and verifying truth than it is about adding big numbers in the shortest time possible. In many ways, mathematics is like a puzzle; you must be able to put all the pieces together. This requires problem solving skills and a fun to way to develop these critical problem solving skills is to solve mathematical puzzles. These puzzles retain the same spirit and soul of mathematics but without the formalism shown in school. Since each problem is different, it can be frustrating but most of time, it can be a rewarding and deeply enriching experience when you crack the puzzle.

Problem Solving Guidelines

1. Read the question multiple times. The first time, you should read to understand what the question is asking. Then read again to identify missing or important information. Always make sure you understand what the question before starting

2. Draw and/or sketch diagrams, charts, picture, or other visuals. This helps organize your thought process and point out important information quickly.

3. If a problem seems too difficult to solve all at once. Break it down into smaller more manageable steps. It is then easier to piece them together for a final solution

4. Identify any patterns you may see. Ask questions like "Am I seeing something happen again and again?"

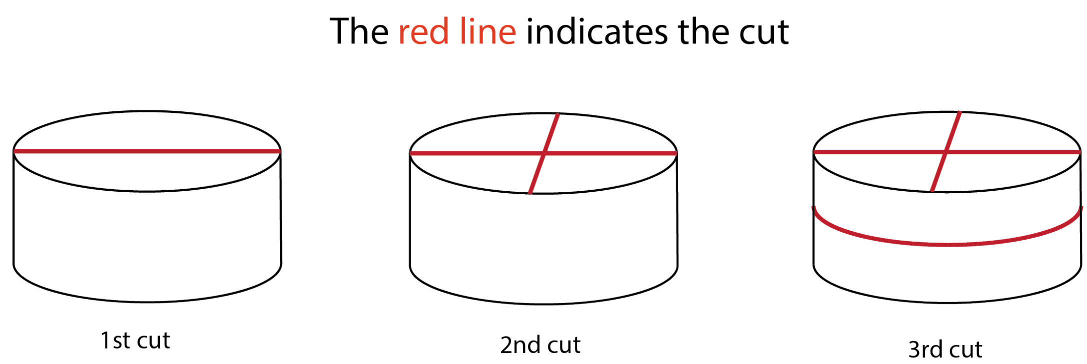
5. It may be helpful to eliminate any impossible (but not **UNLIKELY**) solutions or answers

6. Work backwards. Can I start at the end of the question to help me figure it out? What might possible solutions look like?

7. Sometimes you may have to make a conjecture (a reasonable guess) of what rules they follow and be able to verify if your initial guess was indeed correct.

1. The Birthday Cake

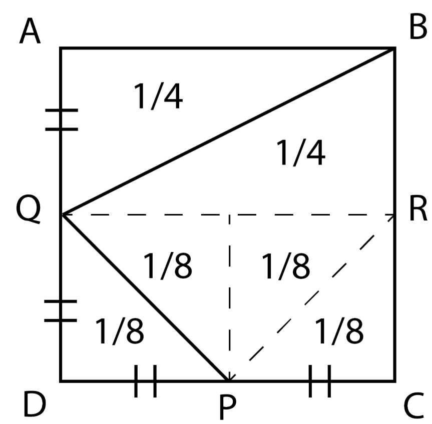
You must make a birthday cake into exactly eight pieces, but you are only allowed 3 straight cuts. How can this be done? Draw your solution below.



2. The Area of the Square

In square ABCD, P is the midpoint of DC and Q is the midpoint of AD. If the area of the quadrilateral QBCP is 15, what is the area of square ABCD?

Solution: Draw line segment QR parallel to DC, as in the following diagram. This segment divides square ABCD into two halves. Since triangles ABQ and RQB are congruent, each is half of rectangle ABRQ and therefore one quarter of square ABCD. Draw line segment P S parallel to DA, and draw line segment P R. Triangles P DQ, P SQ, P SR and P CR are congruent. Therefore each is one quarter of rectangle DCRQ and therefore one eighth of square ABCD.



Quadrilateral QBCP therefore represents $\frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{5}{8}$ if square ABCD. Its area is therefore $\frac{5}{8}$ of the area of the square. Therefore $\frac{5}{8}$ of the area of the square is equal to 15. Hence we have that $\frac{1}{8}$ of the area of the square is equal to 3. We can conclude that the square has an area of 24.

3. The Insurance Salesman

An insurance salesman walk up to house and knocks on the door. A woman answers, and he asks her how many children she has and how old they are. She says I will give you a hint. If you multiply the 3 children's ages, you get 36. He says this is not enough information. So she gives a him 2nd hint. If you add up the children's ages, the sum is the number on the house next door. He goes next door and looks at the house number and says this is still not enough information. So she says she'll give him one last hint which is that her oldest of the 3 plays piano.

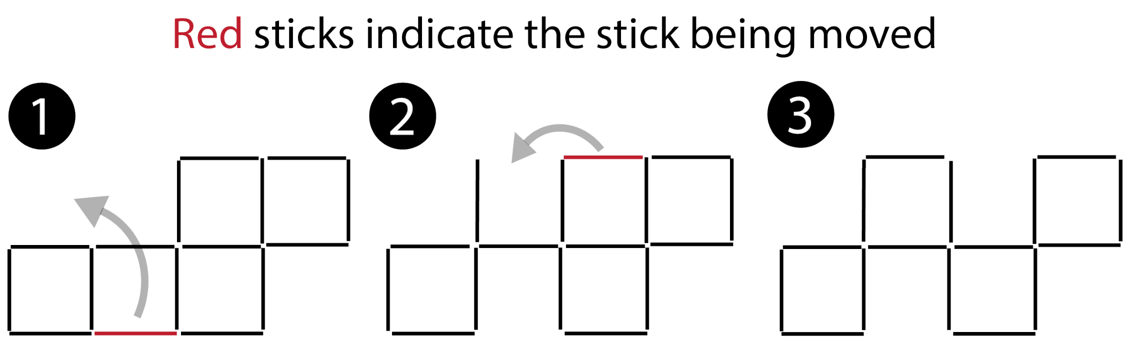
Solution: First, we need to list all the three number combinations that multiply to to 36. We will also add the three numbers to find all the possible sums.

$$1\ 1\ 36 = 38$$
$$1\ 2\ 18 = 21$$
$$1\ 3\ 12 = 16$$
$$1\ 4\ 9 = 14$$
$$6\ 6\ 1 = 13$$
$$2\ 2\ 9 = 13$$
$$2\ 3\ 6 = 11$$
$$3\ 3\ 4 = 10$$

Since the number on the house next door is not enough information, there must be more than one possible combination that add up to the same number as the house next door. For example, if the house next door was 38, then looking up at the chart above, we can see that there is only one sum combination that adds up to 38, and the salesman would immediately deduce that the woman has two children that are 2 years old and one that is 36 years old. Since this is not the case, the only possible sum that appears multiple times is 13 with two combinations of 6 6 1 and 2 2 9. As the final hint, the woman says that their oldest child plays piano, implying that there is an oldest so 6 6 1 can not be the possibility. This leaves 2 2 9 as the only possibility, and hence the woman has two kids that are 2 years old and one that is 9 years old.

4. From 5 squares to 4 Squares

Below you see are 5 squares. Move **two and only two toothpicks** to form 4 squares each with a side length of 1 toothpick. No other shapes are permitted.



5. KenKen Puzzles

In KenKen, the goal is to fill the whole grid with numbers but it must satisfy the following conditions

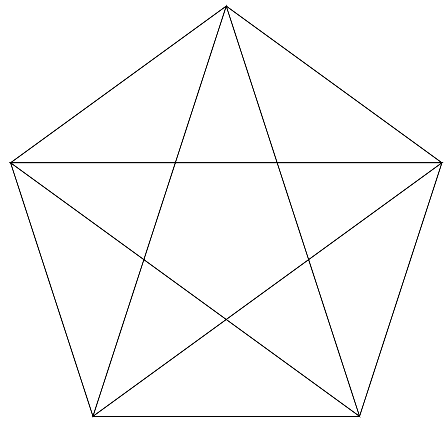
1. In a grid with side length of n, you may only use numbers from 1 to n. For example if you had a 4 by 4 grid, you may only use numbers 1-4
2. No number can be repeated in any row or column
3. Areas with bolded outlines are called "cages". The top left corner of each cage has a target number and an operation. Numbers in the cage must combine (in any order) using the given mathematical operation (+, -, × ÷) to produce the target number

1- 2	3 3	16× 4	1
3	2÷ 2	1	4
6+ 4	1	6× 3	2
1	2÷ 4	2	3 3

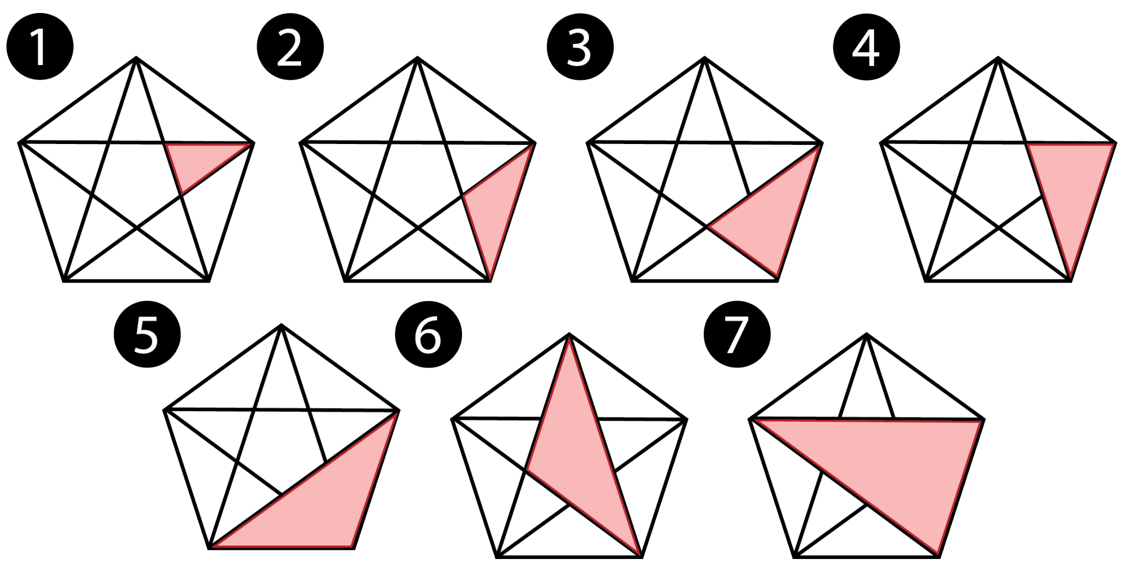
6× 2	4 4	2- 5	3	5- 1	6
3	18× 6	2÷ 4	7+ 2	5	1 1
1	3	2	120× 5	6	4
1- 4	3+ 2	16+ 6	3÷ 1	3	10+ 5
5	1	3	6	10+ 4	2
11+ 6	5	1	4	2	3

6. Counting Triangles

In the pentagon below, how many different triangles are there?



Solution: There are number of solutions students could have used. They could have exhausted all possibilities or more cleverly grouped seven groups of congruent (same shape, same size) triangles. Each group of triangles consists of 5 triangles rotated by exactly $72°$. The seven group are down below. Since there are 7 groups of 5. In total there are $7 \times 5 = 35$ triangles



7. Divisibility

A teacher wrote a large number on the board and asked students to determine what are the divisors of this number.

The first student said, "the number is divisible by 2."
The second student said, "the number is divisible by 3."
the third student said, "the number is divisible by 4."

$$\vdots$$

The 30th students said, "the number is divisible by 31"

The teacher then said that exactly two students, who spoke consecutively were incorrect. Which students spoke incorrectly?

Solution: Since the two numbers are consecutive, one of them is even and the other is odd. Let's say the two numbers were 2 and 3. If a number was not divisible by 2 and 3, then it would also not be divisible by every even number from 2 to 30 and every multiple of 3 from 3 to 30. Thus we have more than just two consecutive students who spoke wrong which contradicts what the teacher said. So, the two numbers cannot be 2 and 3

Using the same logic, we can conclude eliminate 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, and 15.

Could it be, say 21 and 22? That will not work, because we know that it's divisible by 7 and 3 and 2 an 11, so it must be divisible by 21 and 22. Similarly, since we know the number is divisible by all the whole numbers from 2 through 15, it must also be divisible by 18, 20, 21, 22, 24, 26, 28, and 30.

This leaves the following number pair: 16 and 17. Hence the two students who spoke wrong are the 15th and 16th students

8. A Big Number

Using the digits from 1 to 9, two numbers must be made. The product of these two numbers should be as large as possible. What are these numbers?

- Difference of two numbers should be at a minimum
- Higher value digits be placed in the left most position i.e., if A is a 5 digit number and assume it's value to be A=abcde we need to have $a > b > c > d > e$ Then numbers are and 96521 and 8743

9. The Potato Paradox

You have 100 lbs of potatoes, which are 99% water by weight. You let them dehydrate until they are 98% Water. How much do they weigh now?

Hint: Take your time, the answer for most people is rather surprising!

Method 1: Ratio Reasoning

In 100 lbs of potatoes, 99% of it is water which means that there's 99 lbs of water, and 1 lb of solids. It's a 1:99 ratio. If the water decreases to 98%, then the solids account for 2% of the weight. The 2:98 ratio reduces to 1:49. Since the solids still weigh 1 lb, the water must weigh 49 lbs.

Method 2: Algebraic Method

After evaporating water, we have x quantity that contains 1 lbs of potato and $\frac{98}{100}x$ water.

So the total weight can be expressed as:

$$1 + \frac{98}{100}x = x$$
$$1 = x - \frac{98}{100}x$$
$$1 = \frac{2}{100}x$$
$$\frac{100}{2} = x$$
$$x = 50$$

Hence we have that potatoes weigh 50 lbs.

10. The Circular Track

On a circular track, Alphonse is at point A and Beryl is diametrically opposite at point B. Alphonse runs counterclockwise and Beryl runs clockwise. They run at constant, but different, speeds. After running for a while they notice that when they pass each other it is always at the same three places on the track. What is the ratio of their speeds?

Since Alphonse and Beryl always pass each other at the same three places on the track and since they each run at a constant speed, then the three places where they pass must be equally spaced on the track. In other words, the three places divide the track into three equal parts. We are not told which runner is faster, so we can assume that Beryl is the faster runner. Start at one place where Alphonse and Beryl meet. (Now that we know the relative positions of where they meet, we do not actually have to know where they started at the very beginning.) To get to their next meeting place, Beryl runs farther than Alphonse (since she runs faster than he does), so Beryl must run $\frac{2}{3}$ of the track while Alphonse runs $\frac{1}{3}$ of the track in the opposite direction, since the meeting placed are spaced equally at 1 3 intervals of the track. Since Beryl runs twice as far in the same length of time, then the ratio of their speeds is 2 : 1.

11. Bridge Crossing

A group of four people has to cross a bridge. It is dark and the only possible way to cross the bridge safely is to light the path with a flashlight. Unfortunately, the group has only one flashlight. Only two people can cross the bridge at the same and different members of the group takes a different time to cross the bridge.

- Alice crosses the bridge in 1 minute
- Bob crosses the bridge in 2 minutes
- Charlie crosses the bridge in 5 minutes
- Dorothy crosses the bridge in 8 minutes

How can the group cross the bridge in the minimum amount of time possible?

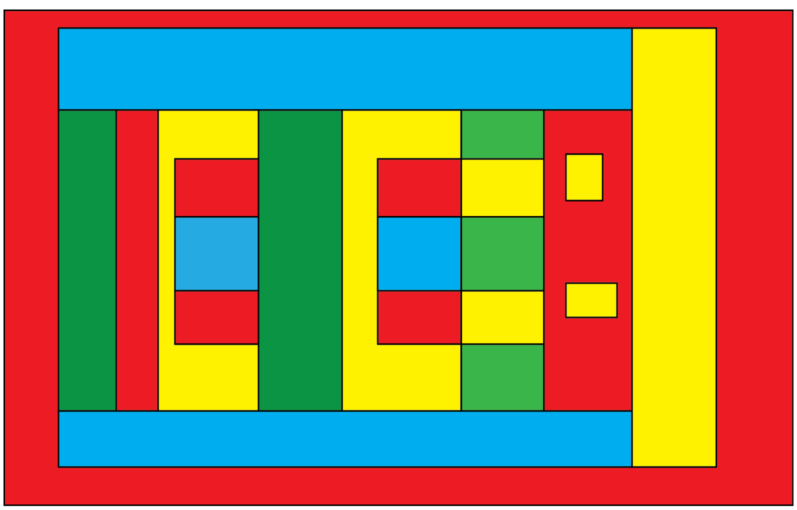
Solution: The order of how people should move across the bridge

Elapsed Time	Action	Starting Side	Ending Side
2 minutes	Alice and Bob cross forward, taking 2 minutes	Charlie, Dorothy	Alice, Bob
3 minutes	Alice returns, taking one minute	Alice, Charlie, Dorothy	Bob
13 minutes	Charlie and Dorothy cross forward, taking 8 minutes	Alice	Bob, Charlie, Dorothy
15 minutes	Bob returns, taking 2 minutes	Alice, Bob	Charlie, Dorothy
17 minutes	Alice and Bob cross forward,		Alice, Bob

12. Four Colouring Theorem To create maps that are easy to read, mapmakers often color them according to a rule that touching regions must always be colored differently. To color a large, complicated map this way, you might think you'd need to use a lot of different colors. But in fact, it has been proven mathematically that you never need more than four colors, no matter what the map looks like. Francis Guthrie made this conjecture in 1852, but it remained unproven until 1976, when Wolfgang Haken and Kenneth Appel showed that it was true!

Also, quite interestingly, this proof required the assistance of a computer to check 1,936 different cases that every other case can be reduced to! To date no one knows a quick short proof of this theorem.

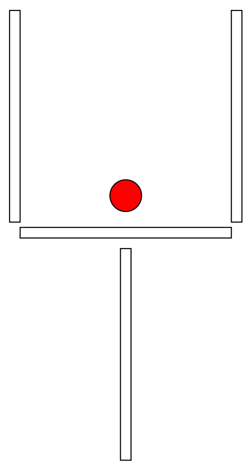
The drawings below aren't maps, but the same principle applies to them. Can you find a way to color all the regions in each drawing, using no more than four different colors, so that regions of the same color never touch (except at corners)?
Hint: Before coloring a pattern, plan how you will do it by penciling in the names of your chosen colors in each region.

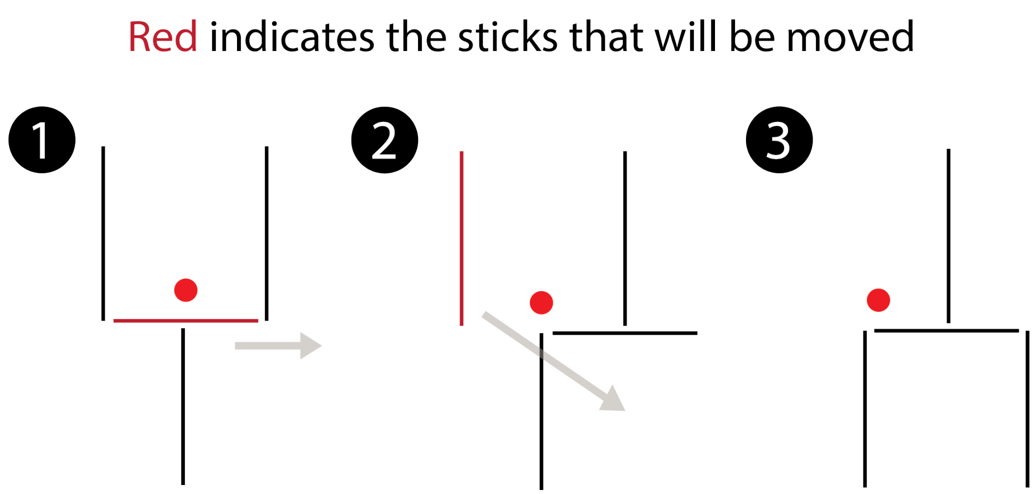


Problem Set

1. The Cherry in the Glass

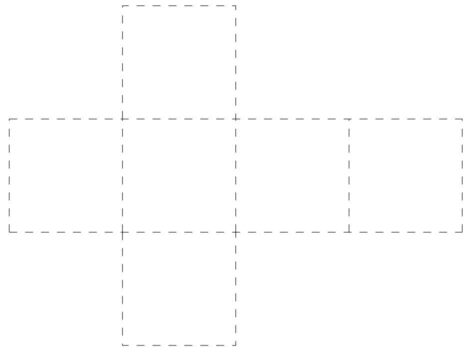
In the image below, we have a cherry in a glass bottle. The glass bottle is composed of 4 sticks. Can you, by moving 2 matchsticks only and not touching the cherry, re-make the glass with the cherry outside?



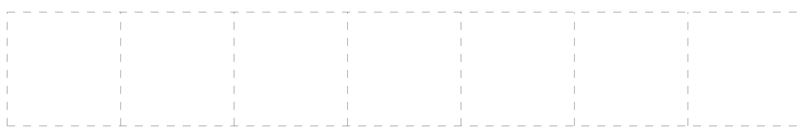


2. From Strip to Cube

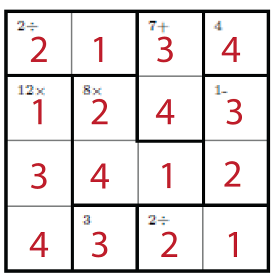
In school, you have seen fold two-dimensional sheets of paper called nets to form 3-D shapes. Traditionally, a cube can be folded by the net shown below.



It turns out that you can fold a cube from a strip of paper as well! The minimum strip that can be folded into a cube is one unit wide and seven units long (1×7). Show how can this be done.

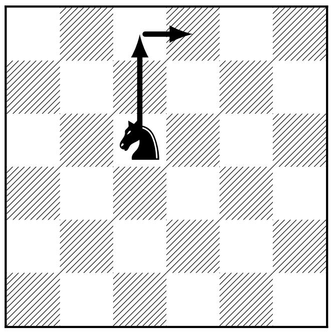


3. More Ken Ken Puzzles



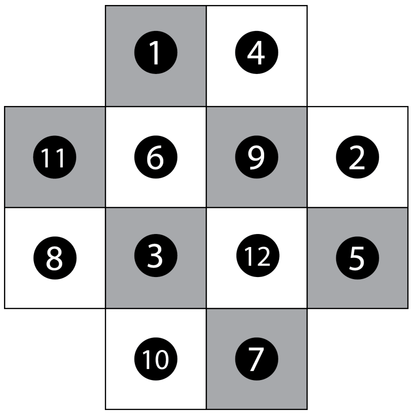
4. The Knight's Tour

One of the most well-known historical math problems is the "The Knight's Tour". In Chess, the knight can only move in an L-shape as shown on diagram on the right.



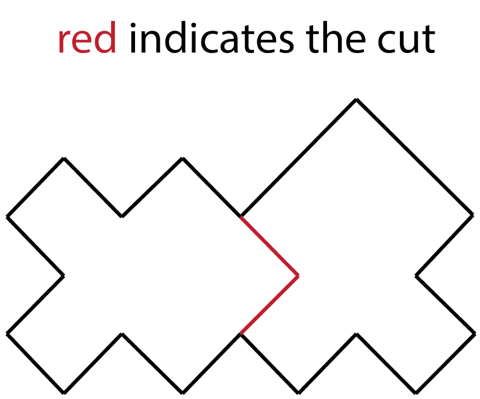
Starting from any square on the chessboard, the objective of the knight's tour is to visit every square every on the board ending at the knight's initial square on the last move. A standard chess board with 8 x 8 board is beyond the scope of Math Circles, so we investigate knight's tour problem on different sizes than the usual 8 x 8 block.

Find the path the knight on the board below so that it visits every square exactly once and on it's last move it's original square.



5. Crazy Cut

Divide the figure below into two using one cut.



6. Missing Dollar Riddle
This is a well known problem that highlights fallacies behind human reasoning.

Three people check into a hotel. The clerk says that the bill is \$30, so each guest pays \$10. Later the clerk realizes the bill should only be \$25. The clerk plans on returning \$5 to the guest, but on his way to their room, he realizes that he can not divide \$5 equally into 3. Since the guests were unaware of the revised bill, he decides to give each guest \$1 back and keep the remaining \$2 for himself as tip. Each guest got \$1 back, so now each guest paid only \$9 with a total amount of \$27 among the three of them. The clerk has \$2. However, if we add the total amount paid with the amount clerk kept for himself, we have \$27 + \$2 = \$29. If the guests originally paid \$30, what happened to the remaining \$1?

Solution: This is a well known problem that highlights fallacies behind human reasoning.

There is no missing dollar. The men spent \$27. Of which \$25 went on the room and \$2 went to the light fingered bell boy. You can not add the \$2 the clerk has to the \$27 the men spent as this \$2 actually comes out of the \$27 the men spent, the other \$25 going on the room.

Put another way consider how the \$30 breaks down:
\$25 - room
\$3 ($3 \times$ \$1) back to the men
\$2 clerk

7. Notable Number
There is a notable number of ten digits, with the following properties **starting from the right**:
All digits from 0 to 9 appear only once in this number
The first digit is divisible by 1
The number formed by the first 2 digits is divisible by 2
The number formed by the first three digits is divisible by 3
The number formed by the first four digits is divisible by 4
The number formed by the first five digits is divisible by 5
The number formed by the first six digits is divisible by 6
The number formed by the first seven digits is divisible by 7
The number formed by the first eight digits is divisible by 8
The number formed by the first nine digits is divisible by 9
The number formed by the ten digits is divisible by 10.

Solution: A number is divisible by 10 if it ends in 0, so the 10th digit ends in 0. Similarly, a number is divisible by 5 if it ends in 5 or 0. Since 0 is used, the 5th digit must be 5

A number is divisible by 2, so the 2nd digit can end in 2, 4, 6, or 8

A number is divisible by 3 if the sum of it's digit is divisible by 3

Below are all possibilities so far for the first three digits.

123	723	147	183	783
129	729	741	189	789
321	921	369	381	981
327	927	963	387	987

A number is divisible by 4 if it's last two digits are divisible by 4. Since the previous number is odd, the fourth digit which is even must end in either 2 or 6. Otherwise, the last two digits won't be divisible by 4.

We can then extend all the possibilities for he first four numbers.

1236	9216	3692	3812	7892
1296	9276	9632	3816	7896
3216	1472	1832	3872	9812
3276	1476	1836	3876	9816
7236	7412	1892	7832	9872
7296	7416	1896	7836	9876

The sixth digit is divisible by 3, if it's divisible by 2 and 3. The first 3 digits is already divisible by 3, so the last 3 digits must be divisible by 3 as well. The only two possible combinations for the fourth, fifth, and sixth digits to be divisible by 3 are

258 and 654

Combining with what we know about the first 4 digits we get

123654	723654	147258	183654	783654
129654	729654	741258	189654	789654
321654	921654	369258	381654	981654
327654	927654	963258	387654	987654

Eighth digit A number is divisible by 8 if either:

- If the hundreds digit is even and last the last two digits is divisible by 8
- If the hundreds digit is odd, and the last two digits plus 4 is divisible by 8

16   32   56   72   96

We know that the seventh digit has to be odd. Therefore, the number formed by the seventh and eighth digits must be divisible by 8. These are the possible combinations:

Combined with what we already know about the first six digits, this gives the following possibilities for the first eight digits of the requested number

18365472	74125896	18965432	78965432	18965472
98165432	38165472	98165472	14725896	98765432

The number formed by the first seven digits of the requested number must be divisible by 7. For the numbers shown above, this only holds for the number 38165472

For the ninth digit, only the digit 9 remains. Note that every number formed by the digits 1 up to 9 is divisible by 9. A number is divisible by 9 if the sum of its digits is divisible by 9. The sum of the digits 1 up to 9 is 45, which is divisible by 9

Therefore the number is 3816547290

8. The Sum and the Product

A teacher says: "I am thinking of two natural numbers bigger than 1. Try to guess what they are.""

The first student knows their product and the second student know their sum.

First:	"I do not know the sum."
Second:	"I knew that. The sum is less than 14."
First:	"I knew that. However, now I already know the numbers."
Second:	"And so do I."

What were the two numbers?

Solution: The teacher tells them both students both numbers are greater than 1. That means 1 can not one of the possible numbers

1. The first student, although knowing the product, can figure out the sum. If the number can ONLY be decomposed to a product of two numbers other than 1 and itself, then the first student could have deduced the sum by adding those two numbers. For example, the number 15 can be broken down into two possible factor pairs (5,3) and (15, 1). We know that both numbers are greater than 1 leaving (5,3) as the only possible factor pair. We

can add 5 + 3 to figure out that 8 is the sum. The first student, however, tells the second student, he does not know the sum, indicating the product can be not be factored uniquely.

2. The second student KNOWS that the first student does not know the sum so he knows the first student was unable to determine the sum with the information he has. How is this possible? He follows up with that the sum is less than 14. Below is the chart containing every possible way of adding two numbers that yields a sum less than 14

Table 1: Possible Sum Pairs

Sum	**Possible Pairs**
4	(2, 2)
5	(2, 3)
6	(4, 2), (3, 3)
7	(5, 2), (4, 3)
8	(6, 2), (8, 3), (4, 4)
10	(7, 2), (7, 3), (6, 4), (5,5)
11	(9, 2), (8, 3), (7, 4), (6, 5)
12	(10, 2), (9, 3), (8, 4), (7, 5), (6,6)
13	(11, 2), (10, 3), (9, 4), (8, 5), (7, 6)

We can safely eliminate 4 and 5 as possible sums because their sum pairs(2,2) and (2, 3) have a product of 4 and 6 respectively. 4 and 6 can be broken down uniquely to 2×2 and 2×3 so the first student could have deduced the sum. In addition, the second student knew the first student could not have known the sum before they told them. This is only possible if all the sum pairs have a product with more than one possible product pair. For example, in the sum table above, a possible sum pair for 10 is (5,5). (5,5) yields a product of 25, with only one possible product pair. If the sum was indeed 10, then the 2nd student couldn't have known for certain that the 1st student didn't know the sum, since there the product could have possibly been 25 and the 1st student could have figured out the sum.

Applying this logic to different sums, we can eliminate possible sums until we are left with 11 as the only possible sum. The sum pairs of 11 are (9,2), (8,3), (7, 4), and (6, 5) with a products of 18, 24, 28, and 30. The second student than says that sum is less than 14, which the first student replies with I already know. This means that out of the four possible products, if you add up their product pairs, they have to be less than 14. 18 is the only possible such product since $24 = 12 \times 2$, $28 = 14 \times 2$ and $30 = 15 \times 2$, and if you add their product pairs, they end up being more than 14. Hence the two numbers are 9 and 2